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DEFORMATION OF A CYLINDRICAL SHELL IN PERIODIC WAVES

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Abstract

In this paper, we investigate the deflection of a thin shell cylinder, mounted at the sea bottom by periodic waves. The motion of the shell is described by the well known thin shell theory. The harmonic water waves diffracted by the cylinder are described by means of the eigenmode expansion. We then solve the coupled equations for the cylinder using the same set of orthogonal functions. By developing the highest derivatives in series of these eigenmodes, we assure then convergence of our solutions.

Resumé

Nous nous intéressons dans ce papier aux déformations subies par une coque de révolution cylindrique, reposant sur le fond marin et excitée par la houle. Les déformations du cylindre sont décrites par le biais de la théorie des coques. Les solutions pour le potentiel harmonique diffracté par le cylindre sont exprimées grâce aux habituelles fonctions propres en profondeur finie. Nous résolvons alors les équations couplées pour le cylindre en utilisant la même base de fonctions. En développant les dérivées d'ordre les plus élevées sur ces bases de fonctions, on assure ainsi la convergence des séries pour les solutions de déplacement du cylindre.

1 Introduction

In this paper we investigate the influence of periodic water waves on the deflection of a thin flexible shell, with large radius, mounted at the sea bottom at depth h and piercing through the free surface. The fluid is assumed to be inviscid and rotationless, hence the velocity potential obeys the Laplace equation. The motion of the shell is described by the classical theory of thin shells. The equations are then reduced to simplified "two dimensional" equations involving the normal, tangential and longitudinal motion u, v, w of the middle surface. This theory requires that the shell thickness d is small in comparison with shell radius R , and the water depth h . The other classical assumptions can be found in Markus [1]. We will focus on the periodic deflections only and we pay no attention to the static deflection due to hydrostatic pressure. By linearity, this effects can be added.

The harmonic water waves diffracted by a rigid cylinder will be described by means of the well known eigenmode expansion. These modes are solutions of the Laplace equation and fit the boundary conditions at the free surface and the sea bottom. To solve the coupled equations for the refracted potential and the motion of the shell we use the same set of orthogonal functions. Due to the boundary conditions of the variables of motion of the shell this can not lead to a converging series. To assure convergence we expand these highest derivatives, as they occur in the equations of motions, in series of these eigenmodes. The lower order derivatives and the functions themselves follow from these expansions. This gives rise to a set of polynomials with unknown coefficients. Making use of the orthogonality relations of the eigenmodes and the expansion of the polynomials in these modes leads to a set of equations with a sparse block, structured by the orthogonality relation, combined with full rows due to the expansions of the polynomials. The boundary conditions of the variables describing the motion of the shell give an extra set of equations. The final 'square' matrix equation can be solved by a standard method.

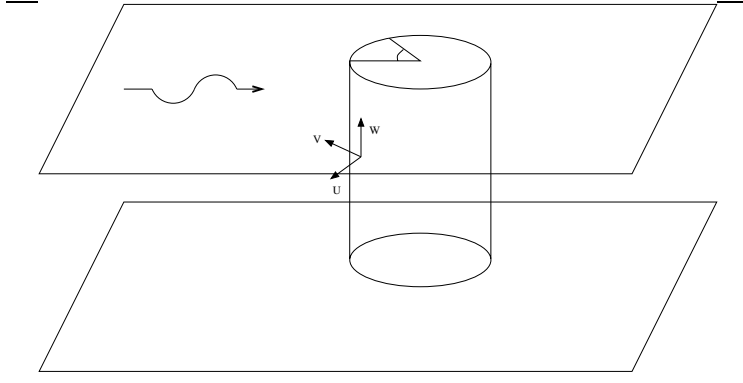
In section 6, an application of this mathematical approach is treated for the case of a flexible beam loaded by a distribution described by the first eigenmode of the water-wave problem. Convergence of the deflection and its derivatives is shown. In the next section, the mathematical model for the shell is given and the method is tested for several values of the flexural rigidity of the shell.

2 Derivation of the main equations

We study the behavior of the thin cylindrical shell in periodic waves in a sea of constant depth h . The shell, emerging from the sea surface is fixed on the sea bottom.

We assume the fluid to be potential and introduce the velocity potential $\mathbf{V} = \nabla\Phi(\underline{x}, t)$ where \mathbf{V} is the fluid velocity vector. We get for the potential $\Phi(\underline{x}, t)$ the Laplace equation $\Delta\Phi = 0$ in the fluid domain.

The linearized free surface $z = 0$, the linearized condition $g\Phi_z + \Phi_{tt} = 0$, at the bottom we have $\frac{\partial\Phi}{\partial z} = 0$ and for the linearized body boundary condition $\frac{\partial\Phi}{\partial n} = u$ where u is the normal deflection of the shell.



Following Love's theory, we derive the equations of motion of the shell.

$$-\nu R \frac{\partial u}{\partial z} + \frac{R}{2}(1+\nu) \frac{\partial^2 v}{\partial z \partial \theta} + R^2 \frac{\partial^2 w}{\partial z^2} + \frac{1}{2}(1-\nu) \frac{\partial^2 w}{\partial \theta^2} - \frac{\rho_0}{E}(1-\nu^2) R^2 \frac{\partial^2 w}{\partial t^2} = 0 \quad (1)$$

$$-\frac{\partial u}{\partial \theta} + \frac{\partial^2 v}{\partial \theta^2} + \frac{R^2}{2}(1-\nu) \frac{\partial^2 v}{\partial z^2} - \frac{\rho_0}{E}(1-\nu^2) R^2 \frac{\partial^2 v}{\partial t^2} + \frac{R}{2}(1+\nu) \frac{\partial^2 w}{\partial z \partial \theta} = 0 \quad (2)$$

$$\begin{aligned} -\chi \left(R^4 \frac{\partial^4 u}{\partial z^4} + 2R^2 \frac{\partial^4 u}{\partial z^2 \partial \theta^2} + \frac{\partial^4 u}{\partial \theta^4} \right) - u - \frac{\rho_0}{E}(1-\nu^2) R^2 \frac{\partial^2 u}{\partial t^2} + \frac{\partial v}{\partial \theta} \\ + \nu R \frac{\partial w}{\partial z} = \frac{(1-\nu^2) R^2}{Ed} P(r, z, \theta, t) \end{aligned} \quad (3)$$

with $\chi = \frac{1}{12} \left(\frac{d}{R} \right)^2$ and P the dynamic pressure due to water waves.

$$P(r, z, \theta, t) = -\rho \frac{\partial \Phi}{\partial t}(r, \theta, z, t) \quad (4)$$

By differentiating equation (1), one can obtain.

$$-\nu R \frac{\partial^2 u}{\partial z^2} + \frac{R}{2}(1+\nu) \frac{\partial^3 v}{\partial z^2 \partial \theta} + R^2 \frac{\partial^3 w}{\partial z^3} + \frac{1}{2}(1-\nu) \frac{\partial^3 w}{\partial z \partial \theta^2} - \frac{\rho_0}{E}(1-\nu^2) R^2 \frac{\partial^3 w}{\partial z \partial t^2} = 0 \quad (5)$$

The differentiation will allow us to expand later this equation in the same eigenfunctions than the two others shell equations. If we choose equation [1] to be zero at $z = -h$ it is automatically full filled everywhere.

We split the potential and the pressure into two parts.

The first ones (ϕ^d and p^d) are due to the diffraction when the shell is rigid. The second ones (ϕ and p) are due to the flexural deflections (radiation).

with

$$P(r, \theta, z, t) = (p^d(r, \theta, z) + p(r, \theta, z)) e^{-i\omega t} \quad (6)$$

$$\Phi(r, \theta, z, t) = (\phi^d + \phi) e^{-i\omega t} \quad (7)$$

$$\begin{aligned}
p^d(r = R) &= \frac{\rho g}{\cosh(kh)} \left(\frac{h + \sigma^{-1} \sinh^2(kh)}{2} \right)^{1/2} f_0(z) \sum_{n=0}^{\infty} \frac{2\epsilon_n(i)^{n+1}}{\pi k R H'_n(kR)} \cos(n\theta) \\
&= \sum_{n=0}^{\infty} A_n \cos(n\theta) \times f_0(z)
\end{aligned}$$

3 Expansion of the solutions in eigenfunctions

We first expand the potential ϕ in the usual eigenfunctions f_m .

$$\phi(r, \theta, z, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} X_{mn} K_n^1(k_m r) \cos(n\theta) f_m(z) e^{-i\omega t} \quad (8)$$

$$+ \sum_{n=0}^{\infty} X_{0n} H_n^1(kr) \cos(n\theta) f_0(z) e^{-i\omega t} \quad (9)$$

Instead of following the standard approach for Sturm-Liouville boundary value problems where expansions in the eigen-function space of the operator is sought, we assume that the fourth derivative of u , the second derivative of v and third derivative of w can be expanded in the same class of eigenfunctions. Then, if we integrate those functions and find for u, v and w

$$\begin{aligned}
u(z, \theta, t) &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty} \frac{u_{mn}}{k_m^4} f_m(z) + \frac{1}{6} C_{1n} (z+h)^3 + C_{2n} \frac{1}{2} (z+h)^2 \right. \\
&\quad \left. + C_{3n} (z+h) + C_{4n} \right\} \cos(n\theta) e^{-i\omega t} \quad (10)
\end{aligned}$$

$$v(z, \theta, t) = \sum_{n=1}^{\infty} \left\{ \frac{v_{0n}}{k^2} f_0(z) - \sum_{m=1}^{\infty} \frac{v_{mn}}{k_m^2} f_m(z) + C_{8n} (z+h) + C_{9n} \right\} \sin(n\theta) e^{-i\omega t} \quad (11)$$

$$\begin{aligned}
w(z, \theta, t) &= \sum_{n=0}^{\infty} \left\{ \frac{w_{0n}}{k^3} g_0(z) - \sum_{m=1}^{\infty} \frac{w_{mn}}{k_m^3} g_m(z) + \frac{1}{2} C_{5n} (z+h)^2 + C_{6n} (z+h) \right. \\
&\quad \left. + C_{7n} \right\} \cos(n\theta) e^{-i\omega t} \quad (12)
\end{aligned}$$

with

$$f_0(z) = \frac{\sqrt{2} \cosh k(z+h)}{(h + \sigma^{-1} \sinh^2(kh))^{1/2}} \quad g_0(z) = \frac{\sqrt{2} \sinh k(z+h)}{(h + \sigma^{-1} \sinh^2(kh))^{1/2}} \quad (13)$$

with k given by the usual dispersion relation $\sigma = k \tanh(kh)$

$$f_n(z) = \frac{\sqrt{2} \cos k_n(z+h)}{(h - \sigma^{-1} \sin^2(k_n h))^{1/2}} \quad g_n(z) = \frac{\sqrt{2} \sin k_n(z+h)}{(h - \sigma^{-1} \sin^2(k_n h))^{1/2}} \quad (14)$$

k_n given by $\sigma = -k_n \tan(k_n h)$

The f_n functions are orthonormal and form a complete set of eigenfunctions for the solution ϕ .

In order to guarantee the convergence of the termwise integrated series to the integral of the expanded function, we have to assume that the series of the successive derivatives converge uniformly. This requirement can be too strong and is not necessary as can be shown by a slightly different approach. However, numerical results confirm the convergence of the fourth derivative of u although slowly. The convergence of the series for u is then guaranteed and fast. The eigenfunctions f_n for ϕ are not eigenfunctions of the mechanical equations. It turns out that the polynomials are convenient to be added, the found solution lies in the proper function space. It fulfils the boundary conditions and in a weak sense the equations. It is reasonable to expect some local convergence problems for the series of the highest derivatives.

4 Boundary conditions

The boundary condition on the cylinder for the potential ϕ in frequency domain is as follow:

$$\frac{\partial \phi}{\partial n} = -i\omega u \quad (15)$$

The boundary condition at the bottom of the fluid domain also represents that the boundary is a rigid wall, therefore

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -h \quad (16)$$

The mechanical boundary conditions at the ends of the shell are then derived by considering that the shell is fixed at $z = -h$ but free at $z = 0$. Following Markus, the eight extra equations read:

$$u = v = w = \frac{\partial u}{\partial z} = 0 \quad \text{for } z = -h \quad (17)$$

$$\frac{\partial w}{\partial z} + \frac{\nu}{R} \left(\frac{\partial v}{\partial \theta} - u \right) = 0 \quad (18)$$

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{R^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{1}{R} \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} = 0$$

$$\frac{\partial}{\partial z} \left[\frac{\partial w}{\partial z} + \nu \left(\frac{1}{R} \frac{\partial v}{\partial \theta} - \frac{u}{R} \right) \right] - \frac{(1-\nu)}{2R} \frac{\partial}{\partial \theta} \left[\frac{1}{R} \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} + 4 \frac{\partial^2 u}{\partial z \partial \theta} \right] = 0$$

We have then to force equation [1] to be zero. Until now, only its derivative with respect to z equals zero. This gives an extra condition for the coefficients C_{8n} and C_{5n} :

$$\frac{nR}{2}(1 + \nu)C_{8n} + R^2C_{5n} = 0 \quad (19)$$

5 Numerical method

For the numerical resolution of the problem, we truncate the sums to N for n , and to M for m . after multiplying each equation by $f_m(z)$, we integrate the equations [5],[2],[3] and [15] from $z = -h$ to $z = 0$ using the orthonormality of these functions. For each equation, we also expand the polynomials in the f_m series.

$$\delta_{0i} = \int_{-h}^0 f_i(z) dz \quad \delta_{1i} = \int_{-h}^0 (z + h)f_i(z) dz \quad (20)$$

$$\delta_{2i} = \int_{-h}^0 (z + h)^2 f_i(z) dz \quad \delta_{3i} = \int_{-h}^0 (z + h)^3 f_i(z) dz \quad (21)$$

This yields, for each $n \in [0 \cdots N]$, to a linear system of $4M + 13$ equations, with unknown values $U_{mn}, V_{mn}, W_{mn}, X_{mn}$ and the C_i coefficients. The force term of the system is logically found to be the pressure due to the diffracted problem for a rigid shell.

6 Numerical results and test of convergence

To make our method clearly understandable, let us first solve, for illustration a simplified case. Let describe the axisymmetric, radial vibration of a cylindrical shell under the symmetric load $q(z)$. The former system of equations can be reduced to the differential equation

$$D \frac{\partial^4 u}{\partial z^4} + \frac{Eh}{R^2} u = q(z) \quad \text{with} \quad q(z) = k f_0(z) \quad (22)$$

We assume that we have the following boundary conditions

$$u = 0 \quad u' = 0 \quad \text{at} \quad z = -h \quad (23)$$

$$u'' = 0 \quad u''' = 0 \quad \text{at} \quad z = 0 \quad (24)$$

the function u writes

$$u(z) = \sum_{m=0}^M \frac{u_m}{k_m^4} f_m(z) + \frac{1}{6} C_1 (z + h)^3 + C_2 \frac{1}{2} (z + h)^2 + C_3 (z + h) + C_4 \quad (25)$$

Choosing $k = 1$, $D = 1$, $h = 50$ and $\frac{Eh}{R^2} = 10^{-2}$, we give the numerical solutions for u , $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ and $u^{(4)}$ obtained for different values of $M(5, 10, 20, 50, 150)$.

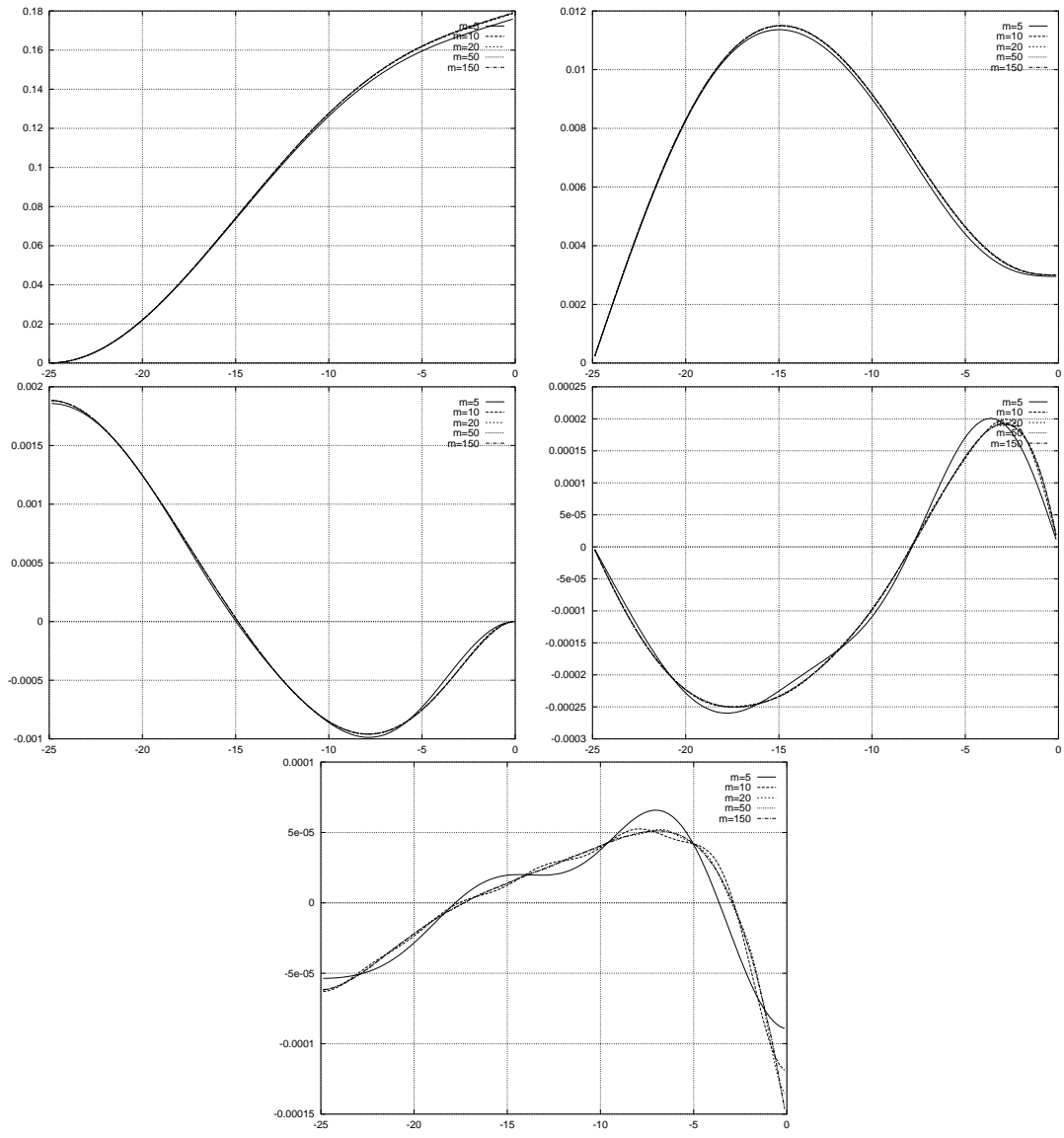


Figure 1: convergence

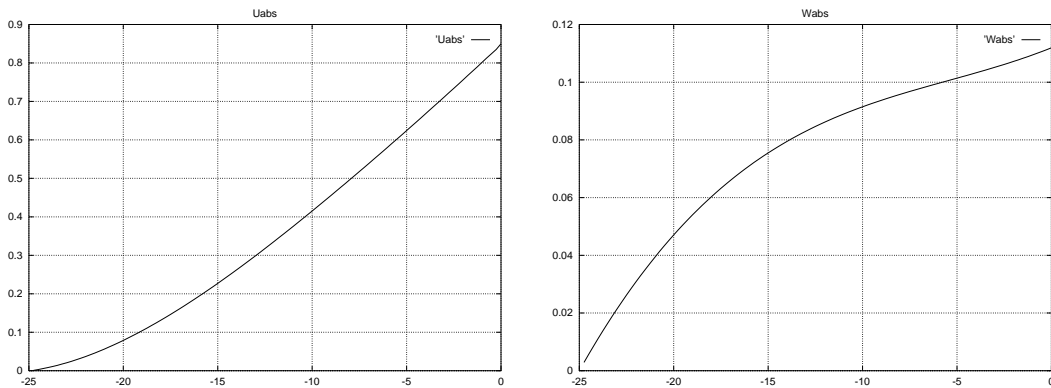


Figure 2: shell deflection

The convergence of u appears very fast and we only need a couple of terms to have a very good approximation of our solution. The series of the fourth derivative seems to have a slow converge. Especially to obtain a right value at the point $z = 0$. This is due to the fact that the f_m functions do not obey the boundary equations of the mechanical problem.

For the complete model it turns out that the convergence of the series for the deflection shows similar behaviour. Depending on the physical parameters we choose the upper bounds of the series. In general the choice $M = 50$ is reasonable. We present here some numerical results for a cylindrical shell with a diameter of 6 m, in a 25m depth sea. The thickness of the shell is 1cm and its density is $2500Kg/m^3$. The rigidity equals $E = 10^9$ and the incoming wave frequency $\omega = 0.3rad/s$. The chosen value for the rigidity is rather low, so the thin structure is very flexible

Examples of the computation of the amplitude of the deflection for u and w against z , for $\theta = 0$ are shown in figure [2] As we suggested we have chosen 50 terms for the series although, more terms are needed to obtain a good convergence of the highest derivatives. In figure [3], we show the cylinder displacement at two different depths ($z=-15m$ and $z=-5m$) and in five different time moments. The main contribution to the displacement is due to the bending of the cylinder.

7 Conclusion

It will be shown that for certain values of the wave frequency the cylinder becomes resonant. This means that our homogeneous boundary value problem has no solution at all. This phenomenon is well known for forced Sturm-Liouville systems with homogeneous boundary conditions. For the structural parameters chosen in our example this happens in the region of the frequency band of a realistic sea-state. If we choose the thickness to be larger, for instance 5cm, the resonance phenomenon occurs at a frequency larger than the frequencies of a sea-spectrum. We also will show results for a larger value of the rigidity parameter. If we take a value for a steel structure we can use a thickness of 1cm leads to no problem in a sea-state. It turns out that this approach can be used for a wide range of parameters

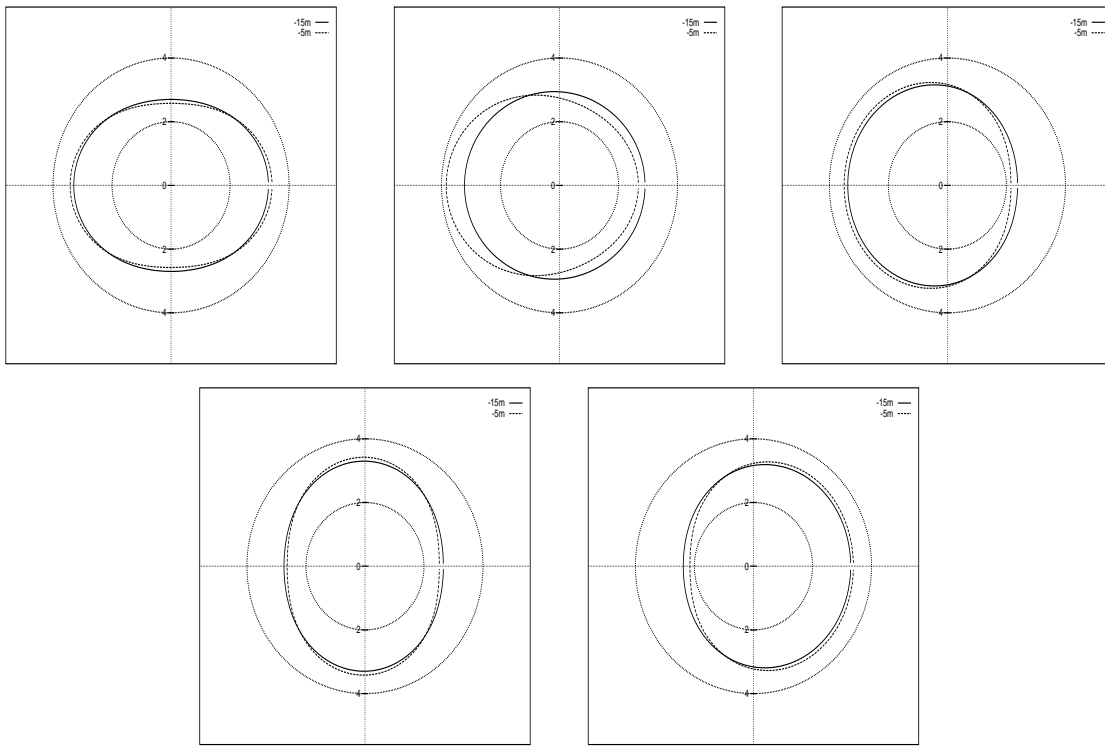


Figure 3: shell deflection

References

- [1] Štefan M̂arkus . The Mechanics of Vibrations of Cylindrical Shells (1988).
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