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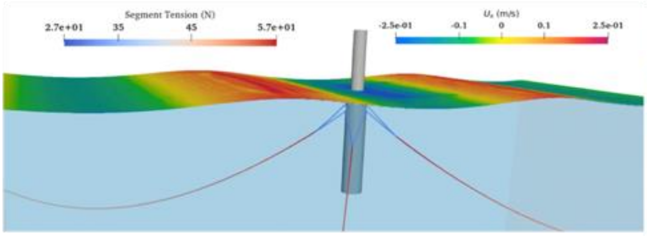

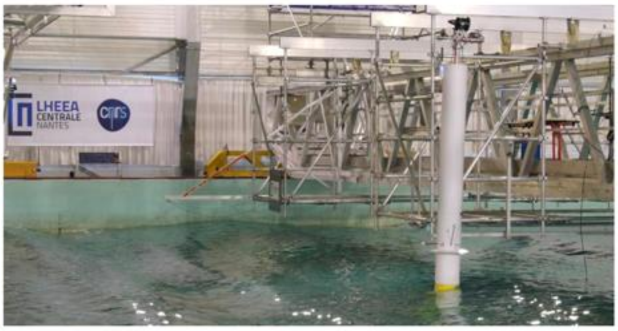

QUELQUES ASPECTS DE LA THEORIES DE DEUXIEME ORDRE POUR L'INTERACTION DES VAGUES AVEC LES CORPS FLOTTANTS

SOME ASPECTS OF THE SECOND ORDER THEORIES FOR INTERACTION OF WATER WAVES WITH FLOATING BODIES

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Résumé

Différentes formulations du problème au deuxième ordre ont été proposé dans la littérature. A cause des différentes notations, description des mouvements, systèmes de coordonnées, points de référence pour le calcul des moments …, les différentes formulations ont l'air assez différentes et quelques malentendus quant à leur interprétation semblent persister. Le but de ce papier est de comparer les différentes formulations avec l'idée qu'il existe une seule théorie de deuxième ordre et toutes les formulations doivent être parfaitement les mêmes.

Summary

There exist different formulations of the 2nd order wave body interaction problem which have been proposed in the past. Due to the different notations, description of motions, referent coordinate systems, reference points for the calculation of the external moments …, the different formulations look sometimes quite different and some misunderstandings in their interpretations seems to persist. The main purpose of this note is to review and compare the different formulations, the basic idea being that there exists only one good formulation and all the different formulations (if they are correct) should be perfectly equivalent

1 Introduction

When formulating the second order problem there are different technical issues to be considered and the following four are probably the most important

1. Description of the body motions
2. Evaluation of the loads
3. Formulation of the Boundary Value Problem (BVP)
4. Solution of the motion equation

2 Description of the body motions

2.1 Few comments on notations

The compact matrix notations are introduced where $[\boldsymbol{A}]$ is used to denote the $n \times m$ matrix with the elements A_{ij} , $(i = 1, n\, ; j = 1, m)$. At the same time, any vector quantity $\boldsymbol{a} = a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k}$ is written as a single column matrix $\{\boldsymbol{a}\}$. Finally, to each vector quantity, the skew symmetric matrix $[\boldsymbol{a}]$ is associated as follows:

$$\{\boldsymbol{a}\} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} \quad , \quad [\boldsymbol{a}] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{1}$$

These notations allow writing the scalar and vector product of two vectors $\{\boldsymbol{a}\}$ and $\{\boldsymbol{b}\}$ as:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \{\boldsymbol{a}\}^T \{\boldsymbol{b}\} \quad , \quad \boldsymbol{a} \wedge \boldsymbol{b} = [\boldsymbol{a}]\{\boldsymbol{b}\} \tag{2}$$

These notations, as well as the description of the nonlinear body dynamics, are inspired by the work of Shabana [6].

2.2 Coordinate systems

Reference is made to Figure 1 where the two coordinate systems are defined:

- (O, x, y, z) Earth fixed inertial coordinate system with arbitrary origin
- (G, x', y', z') Body fixed coordinate system with the origin at the body center of gravity

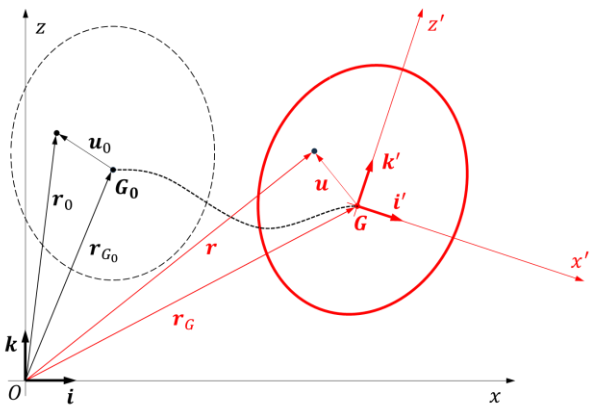


Figure 1: Rigid body motion and the different coordinate systems.

Two sets of coordinates (O, x, y, z) and (G, x', y', z') are related to each other through the transformation matrix $[\boldsymbol{A}]$, so that for any vector quantity $\{\boldsymbol{u}\}$ defined in (O, x, y, z), the following relation is valid:

$$\{\boldsymbol{u}\} = [\boldsymbol{A}]\{\boldsymbol{u}'\} \tag{3}$$

In the case of rigid body we have $\{\boldsymbol{u}'\} = \{\boldsymbol{u}_0\}$ so that:

$$\{\boldsymbol{u}\} = [\boldsymbol{A}]\{\boldsymbol{u}_0\} \tag{4}$$

Finally we also note that in the case of rigid body we have $\{\boldsymbol{n}'\} = \{\boldsymbol{n}_0\}$.

2.3 Nonlinear rigid body motions

The nonlinear rigid body motions are described by the following position, velocity and the acceleration vectors:

$$\{\boldsymbol{r}\} = \{\boldsymbol{r}_G\} + [\boldsymbol{A}]\{\boldsymbol{u}_0\} \tag{5}$$

$$\{\boldsymbol{v}\} = \{\boldsymbol{v}_G\} + [\dot{\boldsymbol{A}}]\{\boldsymbol{u}_0\} = \{\boldsymbol{v}_G\} + [\boldsymbol{\Omega}]\{\boldsymbol{u}\} \tag{6}$$

$$\{\boldsymbol{a}\} = \{\boldsymbol{a}_G\} + [\ddot{\boldsymbol{A}}]\{\boldsymbol{u}_0\} = \{\boldsymbol{a}_G\} + [\dot{\boldsymbol{\Omega}}]\{\boldsymbol{u}\} + [\boldsymbol{\Omega}][\boldsymbol{\Omega}]\{\boldsymbol{u}\} \tag{7}$$

where $\{\boldsymbol{r}_G\}$, $\{\boldsymbol{v}_G\}$ and $\{\boldsymbol{a}_G\}$ are the position, velocity and acceleration of the center of gravity and $\{\boldsymbol{\Omega}\}$ is the instantaneous rotational velocity vector.
The transformation matrix $[\boldsymbol{A}]$ and the instantaneous rotational velocity vector $\{\boldsymbol{\Omega}\}$ are related to each other by the following relation:

$$[\boldsymbol{\Omega}] = [\dot{\boldsymbol{A}}][A]^T \tag{8}$$

2.4 Transformation matrix using the Euler angles

There exists many possibilities to describe the instantaneous position and the orientation of the body. Most common way is to define it by the instantaneous position of the body center of gravity $\{\boldsymbol{r}_G\}$ and the instantaneous rotations angles of the body fixed coordinate system (Euler angles) $\{\boldsymbol{\theta}\}$:

$$\{\boldsymbol{r}_G\} = \begin{Bmatrix} x_G \\ y_G \\ z_G \end{Bmatrix} \quad , \quad \{\boldsymbol{\theta}\} = \begin{Bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} \tag{9}$$

The transformation matrix is obtained by combining the elementary rotations around the different coordinate axis:

$$[\boldsymbol{A}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} \qquad [\boldsymbol{A}_2] = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \qquad [\boldsymbol{A}_3] = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Table 1 :Elementary rotation matrices (roll, pitch and yaw from left to right)

Depending on the choice of the elementary rotation axis and their order of application, there exist 12 different possibilities to define the Euler angles. Here we use the so called zyx or "321" convention which gives:

$$\begin{aligned}[\boldsymbol{A}] &= [\boldsymbol{A}]_{321} = [\boldsymbol{A}_3][\boldsymbol{A}_2][\boldsymbol{A}_1] \\ &= \begin{bmatrix} \cos\theta_z\cos\theta_y & -\sin\theta_z\cos\theta_x + \cos\theta_z\sin\theta_y\sin\theta_x & \sin\theta_z\sin\theta_x + \cos\theta_z\sin\theta_y\cos\theta_x \\ \sin\theta_z\cos\theta_y & \cos\theta_z\cos\theta_x + \sin\theta_z\sin\theta_y\sin\theta_x & -\cos\theta_z\sin\theta_x + \sin\theta_z\sin\theta_y\cos\theta_x \\ -\sin\theta_y & \cos\theta_y\sin\theta_x & \cos\theta_y\cos\theta_x \end{bmatrix}\end{aligned} \tag{10}$$

2.5 Rotational velocity vector

Since the Euler rotations are performed about the instantaneous coordinate axis, the instantaneous rotation velocity vector $\{\boldsymbol{\Omega}\}$ is not simple derivative of the rotation angles but the following relation is valid:

$$\{\boldsymbol{\Omega}\} = [\mathbf{G}]\{\dot{\boldsymbol{\theta}}\} \quad , \quad [\mathbf{G}] = \begin{bmatrix} \cos\theta_z\cos\theta_y & -\sin\theta_z & 0 \\ \sin\theta_z\cos\theta_y & \cos\theta_z & 0 \\ -\sin\theta_y & 0 & 1 \end{bmatrix} \tag{11}$$

2.6 Body kinematics at different orders

The body motion is described by the translation of the body center of gravity $\{\boldsymbol{r}_G\} - \{\boldsymbol{r}_{G_0}\}$ and the instantaneous rotation angles $\{\boldsymbol{\theta}\}$ around the center of gravity. These quantities are developed into the perturbation series up to second order as follows:

$$\{\boldsymbol{r}_G\} - \{\boldsymbol{r}_{G_0}\} = \varepsilon\{\boldsymbol{r}_G^{(1)}\} + \varepsilon^2\{\boldsymbol{r}_G^{(2)}\} \quad , \quad \{\boldsymbol{\theta}\} = \varepsilon\{\boldsymbol{\theta}^{(1)}\} + \varepsilon^2\{\boldsymbol{\theta}^{(2)}\} \tag{12}$$

where $\{\boldsymbol{r}_{G_0}\} = \{x_G^{(0)} \quad y_G^{(0)} \quad z_G^{(0)}\}^T$, see Figure 1.

With these notations, the perturbation series for the different kinematic quantities can be deduced as follows:

- *Transformation matrix* $[\boldsymbol{A}]$

$$[\boldsymbol{A}] = [\,\mathbf{I}\,] + \varepsilon[\boldsymbol{A}^{(1)}] + \varepsilon^2[\boldsymbol{A}^{(2)}] \tag{13}$$

where $[\,\mathbf{I}\,]$ is the identity matrix (all terms zero except the diagonal elements equal to 1).
Whatever the convention used for the definition of the Euler angles, the transformation matrix at first two order takes always the following form:

$$[\boldsymbol{A}^{(1)}] = [\boldsymbol{\theta}^{(1)}] \quad , \quad [\boldsymbol{A}^{(2)}] = [\boldsymbol{\theta}^{(2)}] - \frac{1}{2}([\boldsymbol{H}]_S + [\boldsymbol{H}]_{AS}) = [\boldsymbol{\theta}^{(2)}] - \frac{1}{2}[\boldsymbol{\mathcal{H}}] \tag{14}$$

where $[\boldsymbol{H}]_S$ is the symmetric matrix given by:

$$[\boldsymbol{H}]_S = \begin{bmatrix} \theta_z^{(1)^2} + \theta_y^{(1)^2} & -\theta_y^{(1)}\theta_x^{(1)} & -\theta_z^{(1)}\theta_x^{(1)} \\ -\theta_y^{(1)}\theta_x^{(1)} & \theta_z^{(1)^2} + \theta_x^{(1)^2} & -\theta_z^{(1)}\theta_y^{(1)} \\ -\theta_z^{(1)}\theta_x^{(1)} & -\theta_z^{(1)}\theta_y^{(1)} & \theta_y^{(1)^2} + \theta_x^{(1)^2} \end{bmatrix} \tag{15}$$

The matrix $[\boldsymbol{H}]_{AS}$ is the skew symmetric matrix, which elements depend on the convention which is used. Within the convention adopted here "321" we have:

$$[\boldsymbol{H}]_{AS} = \begin{bmatrix} 0 & \theta_y^{(1)}\theta_x^{(1)} & \theta_z^{(1)}\theta_x^{(1)} \\ -\theta_y^{(1)}\theta_x^{(1)} & 0 & \theta_z^{(1)}\theta_y^{(1)} \\ -\theta_z^{(1)}\theta_x^{(1)} & -\theta_z^{(1)}\theta_y^{(1)} & 0 \end{bmatrix} \tag{16}$$

It is interesting to observe that the symmetric matrix $[\boldsymbol{H}]_S$ can also be written as:

$$[\boldsymbol{H}]_S = -[\boldsymbol{\theta}^{(1)}][\boldsymbol{\theta}^{(1)}] \tag{17}$$

which follows from the fact that:

$$[\boldsymbol{A}]^T[\boldsymbol{A}] = [\,\mathbf{I}\,] \tag{18}$$

- *Rotational velocity vector* $\{\boldsymbol{\Omega}\}$ *and its time derivative* $\{\dot{\boldsymbol{\Omega}}\}$

$$\{\boldsymbol{\Omega}\} = \varepsilon\{\boldsymbol{\Omega}^{(1)}\} + \varepsilon^2\{\boldsymbol{\Omega}^{(2)}\} \quad , \quad \{\dot{\boldsymbol{\Omega}}\} = \varepsilon\{\dot{\boldsymbol{\Omega}}^{(1)}\} + \varepsilon^2\{\dot{\boldsymbol{\Omega}}^{(2)}\} \tag{19}$$

with:

$$\{\boldsymbol{\Omega}^{(1)}\} = \{\dot{\boldsymbol{\theta}}^{(1)}\} \quad , \quad \{\boldsymbol{\Omega}^{(2)}\} = \{\dot{\boldsymbol{\theta}}^{(2)}\} + [\mathbf{G}^{(1)}]\{\dot{\boldsymbol{\theta}}^{(1)}\} \tag{20}$$

$$\{\dot{\boldsymbol{\Omega}}^{(1)}\} = \{\ddot{\boldsymbol{\theta}}^{(1)}\} \quad , \quad \{\dot{\boldsymbol{\Omega}}^{(2)}\} = \{\ddot{\boldsymbol{\theta}}^{(2)}\} + [\dot{\mathbf{G}}^{(1)}]\{\dot{\boldsymbol{\theta}}^{(1)}\} + [\mathbf{G}^{(1)}]\{\ddot{\boldsymbol{\theta}}^{(1)}\} \tag{21}$$

and $[\mathbf{G}^{(1)}]$ is given by:

$$[\mathbf{G}^{(1)}] = \begin{bmatrix} 0 & -\theta_z^{(1)} & 0 \\ \theta_z^{(1)} & 0 & 0 \\ -\theta_y^{(1)} & 0 & 0 \end{bmatrix} \tag{22}$$

- *Normal vector* $\{\boldsymbol{n}\}$, *local displacement* $\{\boldsymbol{r}\} - \{\boldsymbol{r}_0\}$ *and local velocity* $\{\boldsymbol{v}\}$

$$\{\boldsymbol{n}\} = \{\boldsymbol{n}_0\} + \varepsilon\{\boldsymbol{n}^{(1)}\} + \varepsilon^2\{\boldsymbol{n}^{(2)}\} \quad , \quad \{\boldsymbol{r}\} - \{\boldsymbol{r_0}\} = \varepsilon\{\boldsymbol{r}^{(1)}\} + \varepsilon^2\{\boldsymbol{r}^{(2)}\} \quad , \quad \{\boldsymbol{v}\} = \varepsilon\{\boldsymbol{v}^{(1)}\} + \varepsilon^2\{\boldsymbol{v}^{(2)}\} \tag{23}$$

where:

$$\{\boldsymbol{n}^{(1)}\} = [\boldsymbol{A}^{(1)}]\{\boldsymbol{n}_0\} \tag{24}$$

$$\{\boldsymbol{r}^{(1)}\} = \{\boldsymbol{r}_G^{(1)}\} + [\boldsymbol{A}^{(1)}]\{\boldsymbol{u}_0\} \tag{25}$$

$$\{\boldsymbol{v}^{(1)}\} = \{\dot{\boldsymbol{r}}_G^{(1)}\} + [\dot{\boldsymbol{A}}^{(1)}]\{\boldsymbol{u}_0\} \tag{26}$$

$$\{\boldsymbol{n}^{(2)}\} = [\boldsymbol{A}^{(2)}]\{\boldsymbol{n}_0\} \tag{27}$$

$$\{\boldsymbol{r}^{(2)}\} = \{\boldsymbol{r}_G^{(2)}\} + [\boldsymbol{A}^{(2)}]\{\boldsymbol{u}_0\} \tag{28}$$

$$\{\boldsymbol{v}^{(2)}\} = \{\dot{\boldsymbol{r}}_G^{(2)}\} + [\dot{\boldsymbol{A}}^{(2)}]\{\boldsymbol{u}_0\} \tag{29}$$

- *Compact normal vector* $\{\mathbb{N}\}$

The compact normal vector allows for describing both the external forces and moments within the same formalism. It is defined by the following 6 dimensional vector:

$$\{\mathbb{N}\} = \begin{Bmatrix} \{\boldsymbol{n}\} \\ [\boldsymbol{u}]\{\boldsymbol{n}\} \end{Bmatrix} = [\boldsymbol{A}] \begin{Bmatrix} \{\boldsymbol{n}_0\} \\ [\boldsymbol{u}_0]\{\boldsymbol{n}_0\} \end{Bmatrix} \tag{30}$$

where it is understood that the matrix $[\boldsymbol{A}]$ multiplies both the $\{\boldsymbol{n}_0\}$ and $[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}$.
Up to second order we can write:

$$\{\mathbb{N}\} = \{\mathbb{N}^{(0)}\} + \varepsilon\{\mathbb{N}^{(1)}\} + \varepsilon^2\{\mathbb{N}^{(2)}\} \tag{31}$$

where:

$$\{\mathbb{N}^{(0)}\} = \begin{Bmatrix} \{\boldsymbol{n}_0\} \\ [\boldsymbol{u}_0]\{\boldsymbol{n}_0\} \end{Bmatrix} \quad , \quad \{\mathbb{N}^{(1)}\} = [\boldsymbol{A}^{(1)}] \begin{Bmatrix} \{\boldsymbol{n}_0\} \\ [\boldsymbol{u}_0]\{\boldsymbol{n}_0\} \end{Bmatrix} \quad , \quad \{\mathbb{N}^{(2)}\} = [\boldsymbol{A}^{(2)}] \begin{Bmatrix} \{\boldsymbol{n}_0\} \\ [\boldsymbol{u}_0]\{\boldsymbol{n}_0\} \end{Bmatrix} \tag{32}$$

3 External loads

3.1 *Basic principles*

The total external loading is composed of the gravity loading and the pressure loading and we formally write:

$$\{\boldsymbol{\mathcal{F}}\} = \{\boldsymbol{\mathcal{F}}^g\} + \{\boldsymbol{\mathcal{F}}^h\} \tag{33}$$

Most often in the literature, the external loading is expressed relative to the earth fixed coordinate system (O, x, y, z) . In that case the gravity force is constant and of the order $O(1)$. Furthermore, when the reference point for the calculation of the external moment is chosen to be the center of gravity, the moment due to the gravity is zero. That is not the case when the formulation relative to the body fixed coordinate system is used.

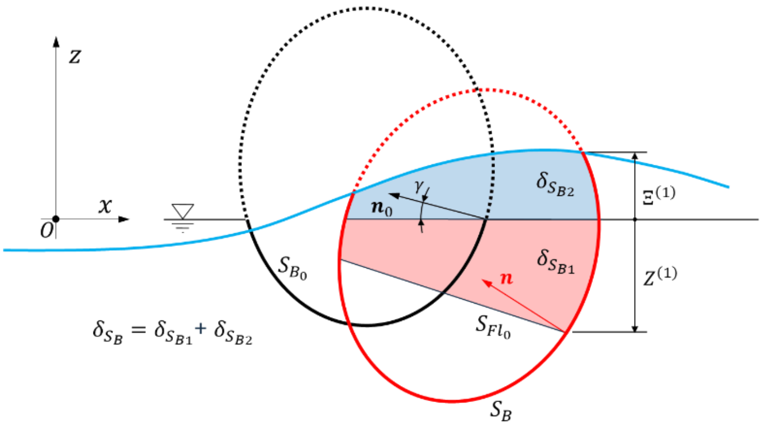


Figure 2: Instantaneous wetted body surface and its separation into the different integration surfaces.

3.2 *Gravity loads*

When expressed relative to the earth fixed coordinate system, the gravity loading is of order $O(1)$ and is given by:

$$\{\boldsymbol{\mathcal{F}}^g\} = -mg \begin{Bmatrix} \{\boldsymbol{k}\} \\ \{\boldsymbol{0}\} \end{Bmatrix} \tag{34}$$

3.3 Nonlinear pressure loads

The fully nonlinear pressure loads are obtained by the integration of the pressure over the instantaneous wetted body surface S_B, and they are written in the compact form as follows:

$$\{\boldsymbol{\mathcal{F}}^h\} = \begin{Bmatrix} \{\boldsymbol{F}^h\} \\ \{\boldsymbol{M}^h\} \end{Bmatrix} = \iint_{S_B} P\{\mathbb{N}\}dS \tag{35}$$

The initial and the instantaneous position and the wetted surface of the body are shown in Figure 2, where the inertial coordinate system (O, x, y, z) is placed at the mean free surface. This coordinate system is sometimes also called the hydrodynamic coordinate system since the fluid pressure is defined with respect to it. As already mentioned, the pressure loads are also expressed in the hydrodynamic coordinate system.

The pressure to be used in (35) is given by the Bernoulli's equation, and is decomposed into its dynamic and the hydrostatic part:

$$P = -\varrho\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + gz\right] = P^d - \varrho gz \tag{36}$$

Similarly, the total pressure loading is also decomposed in two parts as follows:

$$\{\boldsymbol{\mathcal{F}}^h\} = \{\boldsymbol{\mathcal{F}}^{hd}\} + \{\boldsymbol{\mathcal{F}}^{hs}\} \tag{37}$$

with:

$$\{\boldsymbol{\mathcal{F}}^{hd}\} = -\varrho\iint_{S_B}\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2\right]\{\mathbb{N}\}dS \quad , \quad \{\boldsymbol{\mathcal{F}}^{hs}\} = -\varrho g\iint_{S_B} z\{\mathbb{N}\}dS \tag{38}$$

3.4 Pressure at different orders

- *Dynamic pressure component P^d*

$$P^d = -\varrho\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2\right] = \varepsilon P^{(1)} + \varepsilon^2 P^{(2)} \tag{39}$$

with:

$$P^{(1)} = -\varrho\frac{\partial\Phi^{(1)}}{\partial t} \quad , \quad P^{(2)} = -\varrho\frac{\partial\Phi^{(2)}}{\partial t} - \frac{1}{2}\varrho\left\{\nabla\Phi^{(1)}\right\}^2 - \varrho\left(\left\{\boldsymbol{r}^{(1)}\right\}^T\{\nabla\}\right)\frac{\partial\Phi^{(1)}}{\partial t} \tag{40}$$

where $\left\{\boldsymbol{r}^{(1)}\right\}$ is the displacement vector of the point attached to the body (25).

- *Hydrostatic pressure component $-\varrho gz$*

$$-\varrho gz = -\varrho g\{\boldsymbol{r}\}^{\boldsymbol{T}}\{\boldsymbol{k}\} = -\varrho g\left(z^{(0)} + \varepsilon z^{(1)} + \varepsilon^2 z^{(2)}\right) \tag{41}$$

where:

$$z^{(0)} = \quad \{\boldsymbol{k}\}^T\left(\left\{\boldsymbol{r}_{G_0}\right\} + \{\boldsymbol{u}_0\}\right) = z_G^{(0)} + z_0 \tag{42}$$

$$z^{(1)} = \quad \{\boldsymbol{k}\}^T\left\{\boldsymbol{r}^{(1)}\right\} = z_G^{(1)} + \theta_x^{(1)}y_0 - \theta_y^{(1)}x_0 \tag{43}$$

$$z^{(2)} = \quad \{\boldsymbol{k}\}^T\left\{\boldsymbol{r}^{(2)}\right\} = z_G^{(2)} + \theta_x^{(2)}y_0 - \theta_y^{(2)}x_0 - \frac{1}{2}[\mathcal{H}_{31}x_0 + \mathcal{H}_{32}y_0 + \mathcal{H}_{33}z_0] \tag{44}$$

We note that the free surface is defined by $z^{(0)} = 0$ so that we have:

$$z_0|_{S_{F_0}} = -z_G^{(0)} \tag{45}$$

4 Direct formulation of the external loading

4.1 Introduction

First we formally decompose the nonlinear pressure loading (35) into perturbation series up to second order:

$$\{\boldsymbol{\mathcal{F}}^{h}\} = \{\boldsymbol{\mathcal{F}}^{h(0)}\} + \varepsilon\{\boldsymbol{\mathcal{F}}^{h(1)}\} + \varepsilon^2\{\boldsymbol{\mathcal{F}}^{h(2)}\} \tag{46}$$

The direct formulation means that we simply introduce the different perturbation series for the pressure, normal vector and the wetted body surface into the original expression (35) and we collect the terms of different orders of magnitude. This is usually done separately for the pure dynamic pressure contribution and the hydrostatic pressure contribution (38).

In order to be able to decompose the pressure loads at different orders, it is necessary to develop the instantaneous wetted body surface S_B into different orders of magnitude. Reference is made to Figure 2 and the instantaneous wetted body surface is decomposed into its mean component S_{B_0} and the perturbed part δS_B:

$$S_B = S_{B_0} + \varepsilon\left(\delta S_{B_1} + \delta S_{B_2}\right) = S_{B_0} + \varepsilon\delta S_B \tag{47}$$

The surfaces δS_{B_1} and δS_{B_2} are both of the order ε and they are separated here because some authors perform the integration of the hydrostatic part, separately on δS_{B_1} (calm water) and δS_{B_2}. Note that there is no need to look for the higher order terms in the description of the wetted body surface, because the pressure integration over the higher order parts will give the loading contributions of the order higher than ε^2. Furthermore, the integrals over the perturbed wetted body surfaces can be expressed in the form of the line integral over the mean waterline C_{B_0}, as it will be discussed later.

4.2 Pure hydrodynamic contribution

The pure hydrodynamic contribution $\{\boldsymbol{\mathcal{F}}^{hd}\}$, at first two orders, is formally written as:

$$\{\boldsymbol{\mathcal{F}}^{hd}\} = -\varrho \iint_{S_{B_0}+\delta S_B} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2\right]\{\mathbb{N}\}dS = \varepsilon\{\boldsymbol{\mathcal{F}}^{hd(1)}\} + \varepsilon^2\{\boldsymbol{\mathcal{F}}^{hd(2)}\} \tag{48}$$

Within the direct formulation, the integration over the perturbed wetted body surface is performed directly over the total perturbed surface δS_B. Up to second order we get:

$$\{\boldsymbol{\mathcal{F}}^{hd(1)}\} = \iint_{S_{B_0}} P^{(1)}\{\mathbb{N}^{(0)}\}dS \tag{49}$$

$$\{\boldsymbol{\mathcal{F}}^{hd(2)}\} = \iint_{S_{B_0}} \left(P^{(2)}\{\mathbb{N}^{(0)}\} + P^{(1)}\{\mathbb{N}^{(1)}\}\right)dS + \varrho g \int_{C_{B_0}} \Xi^{(1)}\left(\Xi^{(1)} - z^{(1)}\right)\frac{\{\mathbb{N}^{(0)}\}}{\cos\gamma}dC \tag{50}$$

4.3 Hydrostatic contribution

The hydrostatic contribution is given by:

$$\{\boldsymbol{\mathcal{F}}^{hs}\} = -\varrho g \iint_{S_{B_0}+\delta S_B} z\{\mathbb{N}\}dS = \{\boldsymbol{\mathcal{F}}^{hs(0)}\} + \varepsilon\{\boldsymbol{\mathcal{F}}^{hs(1)}\} + \varepsilon^2\{\boldsymbol{\mathcal{F}}^{hs(2)}\} \tag{51}$$

Within the direct formulation of the second order loads, the integration of the hydrostatic pressure over the perturbed wetted body surface is also performed at once i.e. over δS_B directly. The following expressions are obtained at different orders

$$\{\boldsymbol{\mathcal{F}}^{hs(0)}\} = -\varrho g \iint_{S_{B_0}} z^{(0)}\{\mathbb{N}^{(0)}\}dS = \varrho g V \begin{Bmatrix}\{\boldsymbol{k}\}\\ \{\boldsymbol{0}\}\end{Bmatrix} \tag{52}$$

$$\{\boldsymbol{\mathcal{F}}^{hs(1)}\} = -\varrho g \iint_{S_{B_0}} \left(z^{(1)}\{\mathbb{N}^{(0)}\} + z^{(0)}\{\mathbb{N}^{(1)}\}\right)dS \tag{53}$$

$$\{\boldsymbol{\mathcal{F}}^{hs(2)}\} = -\varrho g \iint_{S_{B_0}} \left(z^{(2)}\{\mathbb{N}^{(0)}\} + z^{(0)}\{\mathbb{N}^{(2)}\} + z^{(1)}\{\mathbb{N}^{(1)}\}\right)dS - \frac{1}{2}\varrho g \int_{C_{B_0}} \left[\left(\Xi^{(1)}\right)^2 - \left(z^{(1)}\right)^2\right]\frac{\{\mathbb{N}^{(0)}\}}{\cos\gamma}dC \tag{54}$$

4.4 Total pressure loading

The total pressure loading is obtained by simple summation and the final result is:

$$\{\boldsymbol{\mathcal{F}}^{h(0)}\} = -\varrho g \iint_{S_{B_0}} z^{(0)}\{\mathbb{N}^{(0)}\}dS \tag{55}$$

$$\{\boldsymbol{\mathcal{F}}^{h(1)}\} = \iint_{S_{B_0}} P^{(1)}\{\mathbb{N}^{(0)}\}dS - \varrho g \iint_{S_{B_0}} \left(z^{(1)}\{\mathbb{N}^{(0)}\} + z^{(0)}\{\mathbb{N}^{(1)}\}\right)dS \tag{56}$$

$$\begin{aligned}\{\boldsymbol{\mathcal{F}}^{h(2)}\} = &\iint_{S_{B_0}} \left(P^{(2)}\{\mathbb{N}^{(0)}\} + P^{(1)}\{\mathbb{N}^{(1)}\}\right)dS - \varrho g \iint_{S_{B_0}} \left(z^{(2)}\{\mathbb{N}^{(0)}\} + z^{(0)}\{\mathbb{N}^{(2)}\} + z^{(1)}\{\mathbb{N}^{(1)}\}\right)dS \\ &+\frac{1}{2}\varrho g \int_{C_{B_0}} \left(\Xi^{(1)} - z^{(1)}\right)^2 \frac{\{\mathbb{N}^{(0)}\}}{\cos\gamma} dC\end{aligned} \tag{57}$$

4.5 Total external loading

In order to calculate the total loading the gravity loading (34), needs to be added to the pressure loading. The total external loading at the different orders of magnitude becomes:

$$\{\boldsymbol{\mathcal{F}}^{(0)}\} = \begin{Bmatrix}\{\mathbf{0}\}\\ \{\mathbf{0}\}\end{Bmatrix} \tag{58}$$

$$\{\boldsymbol{\mathcal{F}}^{(1)}\} = \iint_{S_{B_0}} P^{(1)}\{\mathbb{N}^{(0)}\}dS - \varrho g \iint_{S_{B_0}} \left(z^{(1)}\{\mathbb{N}^{(0)}\} + z^{(0)}\{\mathbb{N}^{(1)}\}\right)dS \tag{59}$$

$$\begin{aligned}\{\boldsymbol{\mathcal{F}}^{(2)}\} = &\iint_{S_{B_0}} \left(P^{(2)}\{\mathbb{N}^{(0)}\} + P^{(1)}\{\mathbb{N}^{(1)}\}\right)dS - \varrho g \iint_{S_{B_0}} \left(z^{(2)}\{\mathbb{N}^{(0)}\} + z^{(0)}\{\mathbb{N}^{(2)}\} + z^{(1)}\{\mathbb{N}^{(1)}\}\right)dS \\ &+\frac{1}{2}\varrho g \int_{C_{B_0}} \left(\Xi^{(1)} - z^{(1)}\right)^2 \frac{\{\mathbb{N}^{(0)}\}}{\cos\gamma} dC\end{aligned} \tag{60}$$

It is interesting to note that the first and second order loadings can also be rewritten as:

$$\{\boldsymbol{\mathcal{F}}^{(1)}\} = [\boldsymbol{A}^{(1)}]\{\boldsymbol{\mathcal{F}}^{h(0)}\} + \iint_{S_{B_0}} \left(P^{(1)} - \varrho g z^{(1)}\right)\{\mathbb{N}^{(0)}\}dS \tag{61}$$

$$\{\boldsymbol{\mathcal{F}}^{(2)}\} = [\boldsymbol{A}^{(2)}]\{\boldsymbol{\mathcal{F}}^{h(0)}\} + [\boldsymbol{A}^{(1)}]\{\boldsymbol{\mathcal{F}}^{(1)}\} + \iint_{S_{B_0}} \left(P^{(2)} - \varrho g z^{(2)}\right)\{\mathbb{N}^{(0)}\}dS + \frac{1}{2}\varrho g \int_{C_{B_0}} \left(\Xi^{(1)} - z^{(1)}\right)^2 \frac{\{\mathbb{N}^{(0)}\}}{\cos\gamma} dC \tag{62}$$

5 External forces by other formulations

The direct formulation considers the forces and moments at the same time using the compact notations. Usually in the literature, the forces and the moments are considered separately, and we do the same here.

5.1 *Molin & Marion [3]*

The final expressions for the total pressure forces are:

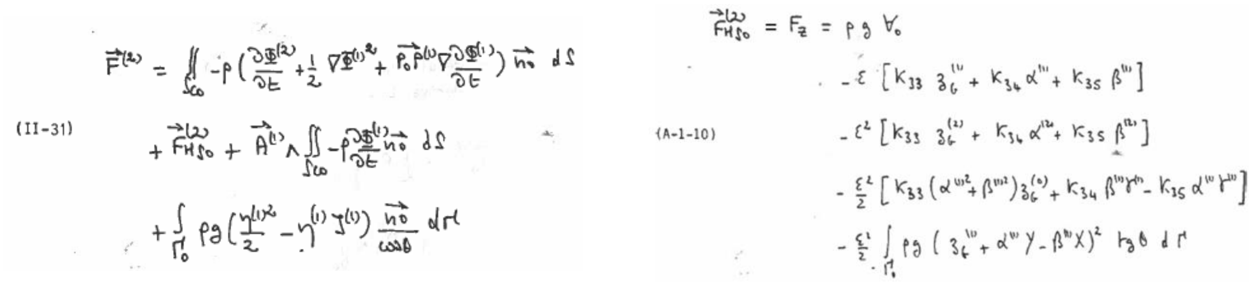


Figure 3: Total pressure force by Molin & Marion (1985).

The description of the nonlinear body motion is such that the antisymmetric part of the second order transformation matrix $\left[\boldsymbol{A}^{(2)}\right]$ is zero i.e. $[\boldsymbol{H}]_{AS} = 0$. The exact derivation of the above expressions for the forces is discussed below using the present notations.

5.1.1 Pure hydrodynamic contribution

The pure hydrodynamic contribution is calculated in exactly the same way as within the direct formulation.

5.1.2 Hydrostatic contribution

The total hydrostatic contribution is first decomposed in two parts:

$$\{\boldsymbol{F}^{hs}\} = \{\boldsymbol{F}^{hs0}\} + \{\boldsymbol{F}^{hs\Xi}\} \tag{63}$$

where $\{\boldsymbol{F}^{hs0}\}$ represents the contribution in calm water and $\{\boldsymbol{F}^{hs\Xi}\}$ is the remaining part:

$$\{\boldsymbol{F}^{hs0}\} = -\varrho g \iint_{S_{B_0}+\delta S_{B1}} z\{\boldsymbol{n}\}dS \quad , \quad \{\boldsymbol{F}^{hs\Xi}\} = -\varrho g \iint_{\delta S_{B2}} z\{\boldsymbol{n}\}dS \tag{64}$$

The contribution $\{\boldsymbol{F}^{hs\Xi}\}$ is of second order and is given by:

$$\left\{\boldsymbol{F}^{hs\Xi(2)}\right\} = -\varrho g \frac{1}{2}\int_{C_{B_0}} \left(\Xi^{(1)}\right)^2 \frac{\{\boldsymbol{n}_0\}}{\cos\gamma} dC \tag{65}$$

The pure hydrostatic contribution in calm water $\{\boldsymbol{F}^{hs0}\}$ is further rewritten as:

$$\begin{aligned}\{\boldsymbol{F}^{hs0}\} = &\quad -\varrho g \iint_{S_{B_0}\pm S_{Fl_0}} z\{\boldsymbol{n}\}dS + \varrho g \iint_{S_{Fl_0}} z\{\boldsymbol{n}\}dS - \varrho g \iint_{\delta S_{B1}} z\{\boldsymbol{n}\}dS \\ = &\quad \varrho g V\{\boldsymbol{k}\} + \varrho g \iint_{S_{Fl_0}} z\{\boldsymbol{n}\}dS - \varrho g \iint_{\delta S_{B1}} z\{\boldsymbol{n}\}dS\end{aligned} \tag{66}$$

where the use of the divergence theorem was made:

$$\iint_{S_{B_0}+S_{Fl_0}} f\{\boldsymbol{n}_0\}dS = -\iiint_V \nabla f dV \tag{67}$$

Up to second order we have:

$$\{\boldsymbol{F}^{hs0}\} = \varrho g V\{\boldsymbol{k}\} + \varepsilon\left\{\boldsymbol{F}^{hs0(1)}\right\}_{S_{Fl_0}} + \varepsilon^2\left(\left\{\boldsymbol{F}^{hs0(2)}\right\}_{S_{Fl_0}} + \left\{\boldsymbol{F}^{hs0(2)}\right\}_{\delta S_{B1}}\right) \tag{68}$$

where:

$$\{\boldsymbol{F}^{hs0(1)}\}_{S_{Fl_0}} = \varrho g \iint_{S_{Fl_0}} z^{(1)}\{\boldsymbol{k}\}dS \tag{69}$$

$$\{\boldsymbol{F}^{hs0(2)}\}_{S_{Fl_0}} = \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(1)}[\boldsymbol{\theta}^{(1)}]\right)\{\boldsymbol{k}\}dS \tag{70}$$

$$\{\boldsymbol{F}^{hs0(2)}\}_{\delta S_{B1}} = \frac{1}{2}\varrho g \int_{C_{B_0}} \left(\mathrm{z}^{(1)}\right)^2 \frac{\{\boldsymbol{n}_0\}}{\cos\gamma} dC \tag{71}$$

With this in mind the total contribution from the hydrostatic pressure $-\varrho g z$ at second order, becomes:

$$\begin{aligned}\{\boldsymbol{F}^{hs(2)}\} &= \{\boldsymbol{F}^{hs0(2)}\}_{S_{Fl_0}} + \{\boldsymbol{F}^{hs0(2)}\}_{\delta S_{B1}} + \{\boldsymbol{F}^{hs\Xi(2)}\} \\ &= \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(1)}[\boldsymbol{\theta}^{(1)}]\right)\{\boldsymbol{k}\}dS - \frac{1}{2}\varrho g \int_{C_{B_0}} \left[\left(\Xi^{(1)}\right)^2 - \left(\mathrm{z}^{(1)}\right)^2\right] \frac{\{\boldsymbol{n}_0\}}{\cos\gamma} dC\end{aligned} \tag{72}$$

5.1.3 Total pressure forces – Expression 1

The total pressure loading is obtained by summing up the hydrodynamic and the hydrostatic contributions:

$$\{\boldsymbol{F}^{h(1)}\} = \iint_{S_{B_0}} P^{(1)}\{\boldsymbol{n}_0\}dS + \varrho g \iint_{S_{Fl_0}} z^{(1)}\{\boldsymbol{k}\}dS \tag{73}$$

$$\begin{aligned}\{\boldsymbol{F}^{h(2)}\} = &\iint_{S_{B_0}} \left(P^{(2)}\{\boldsymbol{n}_0\} + P^{(1)}\{\boldsymbol{n}^{(1)}\}\right)dS + \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(1)}[\boldsymbol{\theta}^{(1)}]\right)\{\boldsymbol{k}\}dS \\ &+ \frac{1}{2}\varrho g \int_{C_{B_0}} \left(\Xi^{(1)} - z^{(1)}\right)^2 \frac{\{\boldsymbol{n}_0\}}{\cos\gamma} dC\end{aligned} \tag{74}$$

The total second order pressure force can also be rewritten as:

$$\{\boldsymbol{F}^{h(2)}\} = \iint_{S_{B_0}} P^{(2)}\{\boldsymbol{n}_0\}dS + \varrho g \iint_{S_{Fl_0}} z^{(2)}\{\boldsymbol{k}\}dS + [\boldsymbol{\theta}^{(1)}]\{\boldsymbol{F}^{h(1)}\} + \frac{1}{2}\varrho g \int_{C_{B_0}} \left(\Xi^{(1)} - z^{(1)}\right)^2 \frac{\{\boldsymbol{n}_0\}}{\cos\gamma} dC \tag{75}$$

5.1.4 Total pressure force – Expression 2

In Molin & Marion the hydrostatic pressure contribution $\{\boldsymbol{F}^{hs0(2)}\}$ is further developed and written in the following form:

$$\{\boldsymbol{F}^{hs0(2)}\} = \{\boldsymbol{F}^{hs0(2)}\}_{\delta S_{B1}} + \{\boldsymbol{F}^{hs0(2)}\}_{S_{Fl_0}} = -\{\boldsymbol{k}\}\frac{1}{2}\varrho g \int_{C_{B_0}} \left(z^{(1)}\right)^2 \tan\gamma\, dC - \varrho g\{\boldsymbol{k}\} \iint_{S_{Fl_0}} z^{(2)} dS \tag{76}$$

so that the total second order pressure force can be rewritten as:

$$\begin{aligned}\{\boldsymbol{\mathcal{F}}^{h(2)}\} = &\iint_{S_{B_0}} P^{(2)}\{\boldsymbol{n}_0\}dS + [\boldsymbol{\theta}^{(1)}]\{\boldsymbol{\mathcal{F}}^{hd(1)}\} + \varrho g \int_{C_{B_0}} \left[\frac{1}{2}\left(\Xi^{(1)}\right)^2 - \Xi^{(1)} z^{(1)}\right] \frac{\{\boldsymbol{n}_0\}}{\cos\gamma} dC \\ &- \varrho g\{\boldsymbol{k}\}\left[\iint_{S_{Fl_0}} z^{(2)} dS + \frac{1}{2}\int_{C_{B_0}} \left(z^{(1)}\right)^2 \tan\gamma\, dC\right]\end{aligned} \tag{77}$$

which is the expression (II-31) from Molin & Marion (see Figure 3 above).

5.1.5 Comparisons: Direct vs Molin & Marion

In order for the two formulations (57) and (75) to be equivalent, the following identity needs to be satisfied:

$$-\varrho g \iint_{S_{B_0}} \left(z^{(2)}\{\boldsymbol{n}_0\} + z^{(0)}\{\boldsymbol{n}^{(2)}\} + z^{(1)}\{\boldsymbol{n}^{(1)}\}\right)dS = \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(1)}[\boldsymbol{\theta}^{(1)}]\right)\{\boldsymbol{k}\}dS \tag{78}$$

Even if this identity follows directly from the use of the divergence theorem, here we demonstrate it by performing the inverse operation. For that purpose, the same transformation as in Molin & Marion is used, which means that we construct the closed volume $S_{B_0} + S_{Fl_0}$ by adding and subtracting the integral over S_{Fl_0} to the integral on the left hand side of (78). The integral over S_{B_0} on the left-hand side of (78) is denoted by I_{B_0} and we write:

$$I_{F_0} = -\varrho g \iint_{S_{B_0} \pm S_{Fl_0}} \left(z^{(2)}\{\boldsymbol{n}_0\} + z^{(0)}\{\boldsymbol{n}^{(2)}\} + z^{(1)}\{\boldsymbol{n}^{(1)}\}\right) dS = I_{F_0}^1 + I_{F_0}^2 \tag{79}$$

where $I_{F_0}^1$ is the integral over the closed volume and $I_{F_0}^2$ is the integral over the waterplane area.
Since $z^{(0)}$ is zero at S_{Fl_0}, it follows that:

$$I_{F_0}^2 = \varrho g \iint_{S_{Fl_0}} \left(z^{(2)}\{\boldsymbol{n}_0\} + z^{(0)}\{\boldsymbol{n}^{(2)}\} + z^{(1)}\{\boldsymbol{n}^{(1)}\}\right) dS = \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(1)}[\boldsymbol{\theta}^{(1)}]\right)\{\boldsymbol{k}\} dS \tag{80}$$

which is exactly the same as the integral at the left hand side of (78).
This means that, in order for (78) to be valid, $I_{F_0}^1$ should be equal to zero:

$$I_{F_0}^1 = -\varrho g \iint_{S_{B_0} + S_{Fl_0}} \left(z^{(2)}\{\boldsymbol{n}_0\} + z^{(0)}\{\boldsymbol{n}^{(2)}\} + z^{(1)}\{\boldsymbol{n}^{(1)}\}\right) dS = 0 \tag{81}$$

To prove that we evaluate the different terms separately as follows:

$$\iint_{S_{B_0}+S_{Fl_0}} z^{(2)}\{\boldsymbol{n}_0\} dS = -\iiint_V \nabla z^{(2)} dV = -\left(\theta_x^{(2)}\{\boldsymbol{j}\} - \theta_y^{(2)}\{\boldsymbol{i}\} - \frac{1}{2}[\mathcal{H}_{31}\{\boldsymbol{i}\} + \mathcal{H}_{32}\{\boldsymbol{j}\} + \mathcal{H}_{33}\{\boldsymbol{k}\}]\right) V$$

$$\iint_{S_{B_0}+S_{Fl_0}} z^{(0)} \left[\boldsymbol{A}^{(2)}\right]\{\boldsymbol{n}_0\} dS = \left([\boldsymbol{\theta}^{(2)}] - \frac{1}{2}[\boldsymbol{\mathcal{H}}]\right) \iint_{S_{B_0}+S_{Fl_0}} z^{(0)}\{\boldsymbol{n}'\} dS = -\left([\boldsymbol{\theta}^{(2)}] - \frac{1}{2}[\boldsymbol{\mathcal{H}}]\right)\{\boldsymbol{k}\} V$$

$$\iint_{S_{B_0}+S_{Fl_0}} z^{(1)} \left[\boldsymbol{A}^{(1)}\right]\{\boldsymbol{n}_0\} dS = \iint_{S_{B_0}+S_{Fl_0}} z^{(1)} \left[\boldsymbol{\theta}^{(1)}\right]\{\boldsymbol{n}_0\} dS = -\left[\boldsymbol{\theta}^{(1)}\right] \iiint_V \nabla z^{(1)} dV = -\left[\boldsymbol{\theta}^{(1)}\right]\left(\theta_x^{(1)}\{\boldsymbol{j}\} - \theta_y^{(1)}\{\boldsymbol{i}\}\right) V$$

By noting that $[\boldsymbol{\mathcal{H}}]\{\boldsymbol{k}\} = \mathcal{H}_{13}\{\boldsymbol{i}\} + \mathcal{H}_{23}\{\boldsymbol{j}\} + \mathcal{H}_{33}\{\boldsymbol{k}\}$ and after summing up all the contributions, we get:

$$I_{F_0}^1 = \varrho g \iint_{S_{B_0}+S_{Fl_0}} \left(\left[\theta_x^{(1)}\theta_z^{(1)} + \frac{1}{2}(\mathcal{H}_{31} + \mathcal{H}_{13})\right]\{\boldsymbol{i}\} + \left[\theta_y^{(1)}\theta_z^{(1)} + \frac{1}{2}(\mathcal{H}_{32} + \mathcal{H}_{23})\right]\{\boldsymbol{j}\}\right) dS = 0 \tag{82}$$

From the description of the body motion we can deduce the following identities:

$$\mathcal{H}_{23} = -\theta_z^{(1)}\theta_y^{(1)} + H_{23} \quad , \quad \mathcal{H}_{13} = -\theta_z^{(1)}\theta_x^{(1)} + H_{13} \quad , \quad \mathcal{H}_{32} = -\theta_z^{(1)}\theta_y^{(1)} - H_{23} \quad , \quad \mathcal{H}_{31} = -\theta_z^{(1)}\theta_x^{(1)} - H_{13} \tag{83}$$

and it can be concluded that the identity (82) is satisfied.

5.2 Ogilvie [4]

Similar to Molin & Marion, the forces and the moments are considered separately, and first we concentrate on the forces. The total hydrodynamic forces are given by the following expression:

$$
\begin{aligned}
\mathbf{F} = {} & \rho g V \mathbf{k} \\
& - \varepsilon\rho \left\{ \iint_{S_m} \mathbf{n}\phi_{1_t} d\sigma + g A_{WP} (\xi_{31} + y_f \xi_{41} - x_f \xi_{51}) \mathbf{k} \right\} \\
& - \varepsilon^2 \rho \left\{ \iint_{S_m} \left\{ \mathbf{n} [\phi_{2_t} + \frac{1}{2} |\nabla \phi_1|^2 + (\boldsymbol{\xi}_1 + \boldsymbol{\alpha}_1 \times \mathbf{X}) \cdot \nabla \phi_{1_t}] + [\boldsymbol{\alpha}_1 \times \mathbf{n}] \phi_{1_t} \right\} d\sigma \right. \\
& \qquad - \frac{1}{2} g \oint_{C_m} d\ell \; \mathbf{n} \{ \zeta_1^2 - 2\zeta_1 (\xi_{31} + y\xi_{41} - x\xi_{51}) \} \\
& \qquad \left. + g A_{WP} \{ (\xi_{32} + y_f \xi_{42} - x_f \xi_{52}) + \xi_{61} (x_f \xi_{41} + y_f \xi_{51}) \} \mathbf{k} \right\} \\
& \qquad + O(\varepsilon^3) \quad , \qquad (73)
\end{aligned}
$$

Figure 4: Total pressure force by Ogilvie (1984).

Within the present notations we can rewrite the above expression in the following form:

$$\{\boldsymbol{\mathcal{F}}^{h(2)}\} = \iint_{S_{B_0}} P^{(2)}\{\boldsymbol{n}_0\}dS + [\boldsymbol{\theta}^{(1)}]\{\boldsymbol{\mathcal{F}}^{hd(1)}\} + \varrho g \int_{C_{B_0}} \left[\frac{1}{2}\left(\Xi^{(1)}\right)^2 - \Xi^{(1)} z^{(1)}\right]\{\boldsymbol{n}_0\}dC - \varrho g\{\boldsymbol{k}\} \iint_{S_{Fl_0}} z^{(2)} dS \tag{84}$$

When compared to the expression (77) of Molin & Marion, the last term in (77) is missing here:

$$-\frac{1}{2}\varrho g\{\boldsymbol{k}\}\int_{C_{B_0}} \left(z^{(1)}\right)^2 \tan\gamma \, dC \tag{85}$$

Another difference is that the term $1/\cos\gamma$, in the line integral, is also missing.
Knowing that, for the wall-sided bodies, we have:

$$\cos\gamma = 1 \quad , \quad \tan\gamma = 0 \tag{86}$$

it is concluded that the Ogilvie's formulation is valid for the wall-sided bodies only.

5.3 Pinkster [5]

Here again the forces and the moments are considered separately and the following expression is given for the forces:

$$
\begin{aligned}
\overline{F}^{(2)} = & -\int_{WL} \tfrac{1}{2}\rho g \zeta_r^{(1)^2} . \overline{n} . d\ell + \overline{\alpha}^{(1)} \times (M . \ddot{\overline{X}}_g^{(1)}) + \\
& -\iint_{S_0} \{-\tfrac{1}{2}\rho |\overline{\nabla}\Phi^{(1)}|^2 - \rho \Phi_t^{(2)} - \rho(\overline{X}^{(1)} . \overline{\nabla}\Phi_t^{(1)})\}\overline{n} . dS + \\
& -\iint_{S_0} -\rho g X_3^{(2)} . \overline{n} . dS + \overline{\alpha}^{(2)} \times (0, 0, \rho g V)
\end{aligned}
$$

Figure 5: Total pressure force by Pinkster (1980).

Within the present notations this can be rewritten as:

$$\left\{\boldsymbol{F}^{h(2)}\right\} = \iint_{S_{B_0}} P^{(2)}\{\boldsymbol{n}_0\} dS + \varrho g \iint_{S_{Fl_0}} z^{(2)}\{\boldsymbol{k}\} dS + \left[\boldsymbol{\theta}^{(1)}\right]\left\{\boldsymbol{F}^{h(1)}\right\} + \frac{1}{2}\varrho g \int_{C_{B_0}} \left(\Xi^{(1)} - z^{(1)}\right)^2 \{\boldsymbol{n}_0\} dC \tag{87}$$

This expression looks the same as the expression (75) from Molin & Marion, for wall-sided bodies.
However, the definition of the second order vertical displacement is not the same (44), and here it is defined as:

$$z^{(2)} = z_G^{(2)} + \theta_x^{(2)} y_0 - \theta_y^{(2)} x_0 \tag{88}$$

This means that the quadratic component $[\boldsymbol{\mathcal{H}}]$ of the second order transformation matrix $\left[\boldsymbol{A}^{(2)}\right]$ is taken to be zero in Pinkster's formulation. This is an error.

5.4 Chen [1]

Here also the forces and the moments are considered separately, but they are collected at the end into the compact generalized loading vector which includes both the forces and the moments. The notations are however different and the brackets [ ] are used to denote the generalized loading vector. Three coordinate systems are introduced:

- (o', x', y', z') absolute (earth fixed) coordinate system (origin at the free surface)
- (o, x, y, z) coordinate system parallel to absolute with the origin at the initial center of gravity
- (O, X, Y, Z) body fixed coordinate system with the origin at the center of gravity

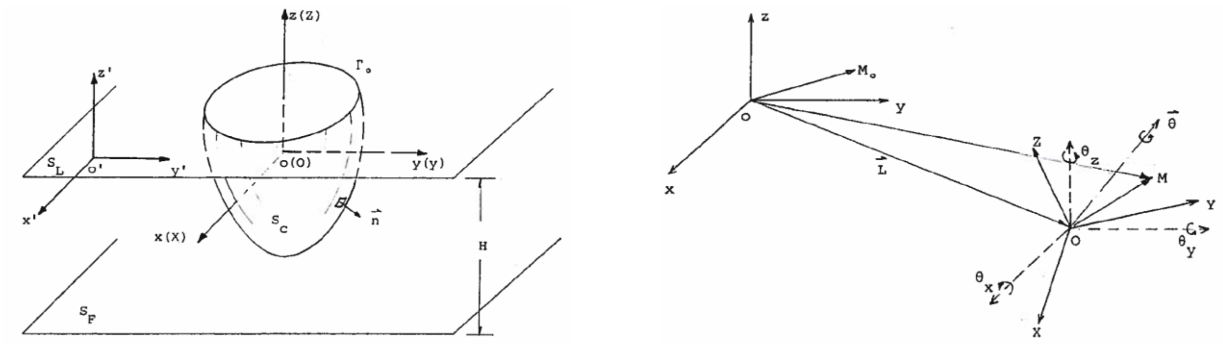


Figure 6: Coordinate systems and body motions in Chen's description (from [1]).

The equation of motion is written with respect to the instantaneous position of the center of gravity. However the external moments are defined with respect to the initial (mean) position of the center of gravity:

$$(4.31) \quad \vec{M}_p(t) = \iint_{S_C} P(M,t)\cdot\vec{r}\wedge\vec{n}\, dS(M) \qquad \vec{r} = \overrightarrow{oM} = \vec{L} + R\cdot\overrightarrow{OM} \tag{89}$$

Before solving the motion equation the moments are transferred to the instantaneous position of the center of gravity. The hydrodynamic loading is decomposed into three components. The first two components are given by the expressions (90) and (91) below:

$$(5.26) \quad [F_{ex1}]^{(2)} = R^{(1)}\cdot[F_I]^{(1)} - \frac{1}{2}\rho g\int_{\Gamma_o}(\eta^{(1)}-\zeta^{(1)})^2\cdot[N]\, d\Gamma + \frac{1}{2}\rho\iint_{S_{CO}}(\nabla\Phi^{(1)})^2\cdot[N]\, dS + \rho\iint_{S_{CO}}\overline{M_oM}^{(1)}\cdot\frac{\partial}{\partial t}\nabla\Phi^{(1)}\cdot[N]\, dS \tag{90}$$

$$(5.27) \quad [F_H]_2^{(2)} = \rho\iint_{S_{CO}}\frac{\partial}{\partial t}\Phi^{(2)}\cdot[N]\, dS \tag{91}$$

where the generalized normal vector $[\mathbf{N}]$ is introduced, in a similar way as in the direct approach, so that we have:

$$[\mathbf{N}] = \begin{Bmatrix} \{\mathbf{n}_0\} \\ [\mathbf{r}_0]\{\mathbf{n}_0\} \end{Bmatrix} = \{\mathbb{N}^{(0)}\} = \begin{Bmatrix} \{\boldsymbol{n}_0\} \\ [\boldsymbol{u}_0]\{\boldsymbol{n}_0\} \end{Bmatrix} \tag{92}$$

Within the present notations the sum of the two components (90) and (91) can be written as:

$$[\mathbf{F}_{\text{ex1}}]^{(2)} + [\mathbf{F}_{\text{H}}]_2^{(2)} = [\boldsymbol{A}^{(1)}]\{\boldsymbol{\mathcal{F}}^{(1)}\} + \iint_{S_{B_0}} P^{(2)}\{\mathbb{N}^{(0)}\}dS + \frac{1}{2}\varrho g\int_{C_{B_0}}\left(\Xi^{(1)} - z^{(1)}\right)^2\frac{\{\mathbb{N}^{(0)}\}}{\cos\gamma}dC \tag{93}$$

At the same time, the loading component which is induced by the hydrostatic pressure, is considered separately and the corresponding expression is given by the expression (94) below:

$$(4.57) \quad [\mathbf{F_S}]^{(2)} = -[\mathbf{K}]\cdot[\mathbf{a}]^{(2)} - [\mathbf{K}]\cdot[\tilde{\mathbf{a}}] + \mathbf{R}^{(1)}\cdot[\mathbf{F_S}]^{(1)} + \rho g\iint_{\in S} z^{(1)}\cdot[\mathbf{N}]\, dS \tag{94}$$

However, it should be noted that the last two terms of $[\mathbf{F}_{\mathrm{S}}]^{(2)}$ in (94) are already included in the expression for $[\mathbf{F}_{\mathrm{ex1}}]^{(2)}$ (90) so that the total second order hydrodynamic loading $\{\boldsymbol{\mathcal{F}}^{(2)}\}$ is written as:

$$\{\boldsymbol{\mathcal{F}}^{(2)}\} = [\mathbf{F}_{\mathrm{ex1}}]^{(2)} + [\mathbf{F}_{\mathrm{H}}]_2^{(2)} + [\,\mathbf{K}\,][\mathbf{a}]^{(2)} + [\,\mathbf{K}\,][\tilde{\mathbf{a}}] \tag{95}$$

In order for (95) to be valid the following identity should be true:

$$[\,\mathbf{K}\,][\mathbf{a}]^{(2)} + [\,\mathbf{K}\,][\tilde{\mathbf{a}}] = -\varrho g \iint_{S_{B_0}} \left(z^{(2)}\{\mathbb{N}^{(0)}\} + z^{(0)}\{\mathbb{N}^{(2)}\}\right) dS = \{\boldsymbol{\mathcal{F}}^K\} = \begin{Bmatrix} \{\boldsymbol{F}^K\} \\ \{\boldsymbol{M}^K\} \end{Bmatrix} \tag{96}$$

where $[\,\mathbf{K}\,]$ is the classical linear hydrostatic restoring matrix and:

$$[\mathbf{a}]^{(2)} = \begin{Bmatrix} \{\boldsymbol{r}_G^{(2)}\} \\ \{\boldsymbol{\theta}^{(2)}\} \end{Bmatrix} \quad , \quad [\tilde{\mathbf{a}}] = \frac{1}{2}\left\{0 \quad 0 \quad -Z_w\left(\theta_x^{(1)^2} + \theta_y^{(1)^2}\right) \quad \theta_z^{(1)}\theta_y^{(1)} \quad -\theta_z^{(1)}\theta_x^{(1)} \quad 0\right\}^T \tag{97}$$

with Z_w denoting the vertical coordinate of the mean free surface in the earth fixed coordinate system which means the initial position of the center of gravity $z_G^{(0)}$ within the present notations.
With the help of the divergence theorem the following expressions can be deduced:

$$\{\boldsymbol{F}^K\} = \varrho g \iint_{S_{Fl_0}} z^{(2)}\{\boldsymbol{k}\} dS \tag{98}$$

$$\{\boldsymbol{M}^K\} = \varrho g \iint_{S_{Fl_0}} z^{(2)}[\boldsymbol{u}_0]\{\boldsymbol{k}\} dS - \varrho g V z_B^{(0)} \left(\begin{Bmatrix} \theta_x^{(2)} \\ \theta_y^{(2)} \\ 0 \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} -\mathcal{H}_{32} \\ \mathcal{H}_{31} \\ 0 \end{Bmatrix} \right) \tag{99}$$

where the fact that, within the Chen's formulation, the matrix $[\boldsymbol{\mathcal{H}}]$ is symmetric, was used and $z_B^{(0)}$ denotes the initial position of the center of buoyancy.
The above expressions are usually rewritten in the following form:

$$\{\boldsymbol{F}^K\} = \begin{Bmatrix} 0 \\ 0 \\ z_G^{(2)}K_{33} + \theta_x^{(2)}K_{34} + \theta_y^{(2)}K_{35} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} 0 \\ 0 \\ z_G^{(0)}\mathcal{H}_{33}K_{33} - \mathcal{H}_{32}K_{34} + \mathcal{H}_{31}K_{35} \end{Bmatrix} \tag{100}$$

$$\{\boldsymbol{M}^K\} = \begin{Bmatrix} z_G^{(2)}K_{34} + \theta_x^{(2)}K_{44} + \theta_y^{(2)}K_{45} \\ z_G^{(2)}K_{35} + \theta_x^{(2)}K_{45} + \theta_y^{(2)}K_{55} \\ 0 \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} z_G^{(0)}\mathcal{H}_{33}K_{34} - \mathcal{H}_{32}K_{44} + \mathcal{H}_{31}K_{45} \\ z_G^{(0)}\mathcal{H}_{33}K_{35} - \mathcal{H}_{32}K_{45} + \mathcal{H}_{31}K_{55} \\ 0 \end{Bmatrix} \tag{101}$$

where the classical restoring coefficients K_{ij} were introduced.
This allows writing the total generalized loading vector $\{\boldsymbol{\mathcal{F}}^K\}$ in the form:

$$\{\boldsymbol{\mathcal{F}}^K\} = [\boldsymbol{K}]\{\boldsymbol{\xi}^{(2)}\} + [\boldsymbol{K}]\{\tilde{\boldsymbol{\xi}}^{(2)}\} \tag{102}$$

where:

$$\{\boldsymbol{\xi}^{(2)}\} = \left\{x_G^{(2)} \quad y_G^{(2)} \quad z_G^{(2)} \quad \theta_x^{(2)} \quad \theta_y^{(2)} \quad \theta_z^{(2)}\right\}^T \quad , \quad \{\tilde{\boldsymbol{\xi}}^{(2)}\} = \frac{1}{2}\left\{0 \quad 0 \quad z_G^{(0)}\mathcal{H}_{33} \quad -\mathcal{H}_{32} \quad \mathcal{H}_{31} \quad 0\right\}^T \tag{103}$$

Knowing that, within the notations used by Chen [1] , we have:

$$\mathcal{H}_{31} = -\theta_z^{(1)}\theta_x^{(1)} \quad , \quad \mathcal{H}_{32} = -\theta_z^{(1)}\theta_y^{(1)} \quad , \quad \mathcal{H}_{33} = \theta_y^{(1)^2} + \theta_x^{(1)^2} \tag{104}$$

it can be concluded that the identity (96) is satisfied.

6 External moments

6.1 Introduction

Within the direct formulation of the external loading, the forces and the moments are considered together using the compact notations:

$$\{\mathcal{F}\} = \begin{Bmatrix} \{\boldsymbol{F}\} \\ \{\boldsymbol{M}\} \end{Bmatrix} \tag{105}$$

which leads to the generalized expressions at each order of magnitude, in relatively simple form (58),(59),(60).
However, most often in the literature the forces and the moments are considered separately, and here we discuss the comparisons for the moments. We also note that the formulation proposed by Chen [1], where the same generalized notations are used is not considered explicitly here.

6.2 Choice of the reference point for the moments

When considering the moments, the first convention to be agreed on, is the reference point for moment's definition. The choice of this point is arbitrary. Since the motion equation is usually written with respect to the instantaneous position of the center of gravity, it seems natural to choose this point as the reference for the definition of the moments. However, some authors use the different points such as: origin of the inertial coordinate system, position of the initial center of gravity or some others. When expressed with respect to the center of gravity, the fully nonlinear pressure induced external moment is given by the following expression:

$$\{\boldsymbol{M}\} = \iint_{S_B} P[\boldsymbol{u}]\{\boldsymbol{n}\}dS \tag{106}$$

Within the direct formulation, the reference point for the definition of the moments is the instantaneous position of the center of gravity.

6.3 Molin & Marion [3]

The final expressions of Molin & Marion's formulation are shown in Figure 7 below.

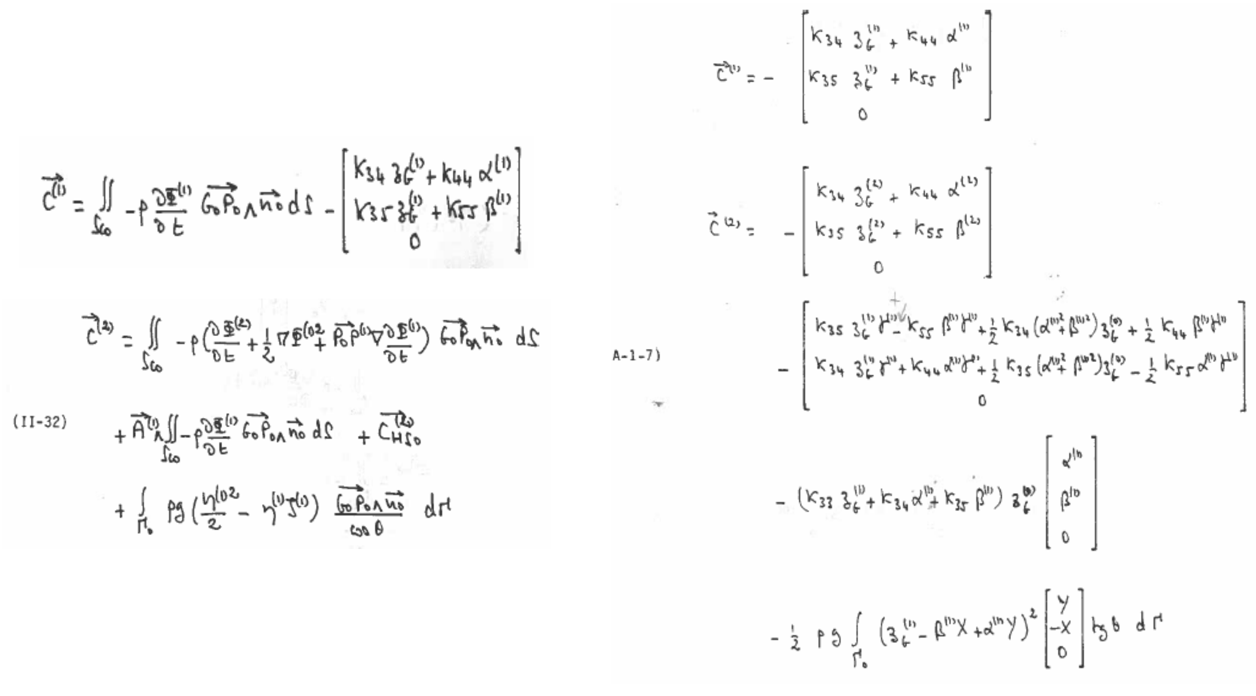


Figure 7: Hydrodynamic pressure moment by Molin & Marion.

The reference point for the definition of the moments is the instantaneous position of the center of gravity. The above expressions are now discussed using the present notations.

6.3.1 Pure hydrodynamic contribution

Similar to the forces, the pure hydrodynamic contribution is exactly the same as in the direct formulation and we recall it here:

$$\{\boldsymbol{M}^{hd(1)}\} = \iint_{S_{B_0}} P^{(1)}[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS \tag{107}$$

$$\{\boldsymbol{M}^{hd(2)}\} = \iint_{S_{B_0}} \left(P^{(2)} + P^{(1)}[\boldsymbol{\theta}^{(1)}]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS + \varrho g \int_{C_{B_0}} \Xi^{(1)}\left(\Xi^{(1)} - z^{(1)}\right)\frac{[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}}{\cos\gamma}dC \tag{108}$$

6.3.2 Hydrostatic contribution

The total hydrostatic contribution is first decomposed in two parts:

$$\{\boldsymbol{M}^{hs}\} = \{\boldsymbol{M}^{hs0}\} + \{\boldsymbol{M}^{hs\Xi}\} \tag{109}$$

where $\{\boldsymbol{M}^{hs0}\}$ represents the contribution in calm water and $\{\boldsymbol{M}^{hs\Xi}\}$ is the remaining part:

$$\{\boldsymbol{M}^{hs0}\} = -\varrho g \iint_{S_{B_0}+\delta S_{B1}} z[\boldsymbol{u}]\{\boldsymbol{n}\}dS \quad , \quad \{\boldsymbol{M}^{hs\Xi}\} = -\varrho g \iint_{\delta S_{B2}} z[\boldsymbol{u}]\{\boldsymbol{n}\}dS \tag{110}$$

The contribution $\{\boldsymbol{M}^{hs\Xi}\}$ is of second order and is given by:

$$\{\boldsymbol{M}^{hs\Xi(2)}\} = -\varrho g \frac{1}{2}\int_{C_{B_0}} \left(\Xi^{(1)}\right)^2 \frac{[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}}{\cos\gamma}dC \tag{111}$$

The pure hydrostatic contribution in calm water $\{\boldsymbol{M}^{hs0}\}$ is further rewritten as:

$$\begin{aligned}\{\boldsymbol{M}^{hs0}\} &= -\varrho g \iint_{S_{B_0}} z[\boldsymbol{u}]\{\boldsymbol{n}\}dS - \varrho g \iint_{\delta S_{B1}} z[\boldsymbol{u}]\{\boldsymbol{n}\}dS \\ &= \varrho g \iint_{S_{Fl_0}} z[\boldsymbol{u}]\{\boldsymbol{n}\}dS - \varrho g \iiint_{V_0} [\boldsymbol{\nabla}](z\{\boldsymbol{u}\})dV_0 - \varrho g \iint_{\delta S_{B1}} z[\boldsymbol{u}]\{\boldsymbol{n}\}dS\end{aligned} \tag{112}$$

where the use of the divergence theorem was made.

The total pure hydrostatic contribution in calm water $\{\boldsymbol{M}^{hs0}\}$ is decomposed in three parts:

$$\{\boldsymbol{M}^{hs0}\} = \{\boldsymbol{M}^{hs0}\}_{S_{Fl_0}} + \{\boldsymbol{M}^{hs0}\}_V + \{\boldsymbol{M}^{hs0}\}_{\delta S_{B1}} \tag{113}$$

where $\{\boldsymbol{M}^{hs0}\}_{S_{Fl_0}}$, $\{\boldsymbol{M}^{hs0}\}_V$ and $\{\boldsymbol{M}^{hs0}\}_{\delta S_{B1}}$ are respectively, the first, second and third term in (112).

- *Component* $\{\boldsymbol{M}^{hs0}\}_{S_{Fl_0}}$

$$\{\boldsymbol{M}^{hs0}\}_{S_{Fl_0}} = \varrho g \iint_{S_{Fl_0}} z[\boldsymbol{u}]\{\boldsymbol{n}\}dS \tag{114}$$

This can be developed as:

$$\{\boldsymbol{M}^{hs0}\}_{S_{Fl_0}} = \varrho g \iint_{S_{Fl_0}} z[\boldsymbol{u}]\{\boldsymbol{n}\}dS = \left(1 + \varepsilon[\boldsymbol{\theta}^{(1)}]\right)\iint_{S_{Fl_0}} \left(\varepsilon z^{(1)} + \varepsilon^2 z^{(2)}\right)[\boldsymbol{u}_0]\{\boldsymbol{k}\}dS \tag{115}$$

so that, at different orders we have:

$$\{\boldsymbol{M}^{hs0(1)}\}_{S_{Fl_0}} = \varrho g \iint_{S_{Fl_0}} z^{(1)}[\boldsymbol{u}_0]\{\boldsymbol{k}\}dS \quad , \quad \{\boldsymbol{M}^{hs0(2)}\}_{S_{Fl_0}} = \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(1)}[\boldsymbol{\theta}^{(1)}]\right)[\boldsymbol{u}_0]\{\boldsymbol{k}\}dS \tag{116}$$

- *Component* $\{\boldsymbol{M}^{hs0}\}_V$

This component can be written as:

$$\{\boldsymbol{M}^{hs0}\}_V = -\varrho g \iiint_{V_0} [\boldsymbol{\nabla}](z\{\boldsymbol{u}\})dV_0 = -\varrho g V z_B^{(0)} \begin{Bmatrix} -A_{23} \\ A_{13} \\ 0 \end{Bmatrix} \tag{117}$$

where A_{13} and A_{23} are the elements of the transformation matrix.
At different orders we have:

$$\{\boldsymbol{M}^{hs0(1)}\}_V = -\varrho g V z_B^{(0)} \begin{Bmatrix} \theta_x^{(1)} \\ \theta_y^{(1)} \\ 0 \end{Bmatrix} \quad , \quad \{\boldsymbol{M}^{hs0(2)}\}_V = -\varrho g V z_B^{(0)} \left(\begin{Bmatrix} \theta_x^{(2)} \\ \theta_y^{(2)} \\ 0 \end{Bmatrix} - \frac{1}{2} \begin{Bmatrix} -\mathcal{H}_{23} \\ \mathcal{H}_{13} \\ 0 \end{Bmatrix} \right) \tag{118}$$

- *Component* $\{\boldsymbol{M}^{hs0}\}_{\delta S_{B1}}$

$$\{\boldsymbol{M}^{hs0}\}_{\delta S_{B1}} = -\varrho g \iint_{\delta S_{B1}} z[\boldsymbol{u}]\{\boldsymbol{n}\}dS \tag{119}$$

This term is of second order and is given by:

$$\{\boldsymbol{M}^{hs0}\}_{\delta S_{B1}} = \frac{1}{2}\varrho g \int_{C_{B_0}} \left(\mathrm{z}^{(1)}\right)^2 \frac{[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}}{\cos\gamma} dC \tag{120}$$

6.3.3 Total pressure induced moment

The total pressure induced moment $\{\mathbf{M}\}$ is obtained by summing up the different contributions

$$\{\boldsymbol{M}\} = \{\boldsymbol{M}^{hd}\} + \{\boldsymbol{M}^{hs\Xi}\} + \{\boldsymbol{M}^{hs0}\}_{S_{Fl_0}} + \{\boldsymbol{M}^{hs0}\}_V + \{\boldsymbol{M}^{hs0}\}_{\delta S_{B1}} \tag{121}$$

The following expressions are deduced:

$$\{\boldsymbol{M}^{(1)}\} = \iint_{S_{B_0}} P^{(1)}[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS + \varrho g \iint_{S_{Fl_0}} z^{(1)}[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS - \varrho g V z_B^{(0)} \begin{Bmatrix} \theta_x^{(1)} \\ \theta_y^{(1)} \\ 0 \end{Bmatrix} \tag{122}$$

$$\begin{aligned} \{\boldsymbol{M}^{(2)}\} = & \iint_{S_{B_0}} \left(P^{(2)} + P^{(1)}[\boldsymbol{\theta}^{(1)}]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS + \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(1)}[\boldsymbol{\theta}^{(1)}]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS \\ & -\varrho g V z_B^{(0)} \left(\begin{Bmatrix} \theta_x^{(2)} \\ \theta_y^{(2)} \\ 0 \end{Bmatrix} - \frac{1}{2} \begin{Bmatrix} -\mathcal{H}_{23} \\ \mathcal{H}_{13} \\ 0 \end{Bmatrix} \right) + \frac{1}{2}\varrho g \int_{C_{B_0}} \left(\Xi^{(1)} - z^{(1)}\right)^2 \frac{[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}}{\cos\gamma} dC \end{aligned} \tag{123}$$

where we note that, for the formulation of Molin & Marion, we have $\mathcal{H}_{23} = -\theta_z^{(1)}\theta_y^{(1)}$ and $\mathcal{H}_{13} = -\theta_z^{(1)}\theta_x^{(1)}$.
It should also be noted that further transformations were made in Molin & Marion (1985) and the expression (123) is rewritten in the different form as shown in Figure 7. We do not do this here.

6.3.4 Comparisons Molin & Marion vs direct

First we recall the expressions to be compared:

- *Direct formulation*

$$\begin{aligned} \{\boldsymbol{M}^{(2)}\} = & \iint_{S_{B_0}} \left(P^{(2)} + P^{(1)}[\boldsymbol{\theta}^{(1)}]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS - \varrho g \iint_{S_{B_0}} \left(z^{(2)} + z^{(0)}[\boldsymbol{A}^{(2)}] + z^{(1)}[\boldsymbol{\theta}^{(1)}]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS \\ & + \frac{1}{2}\varrho g \int_{C_{B_0}} \left(\Xi^{(1)} - z^{(1)}\right)^2 \frac{[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}}{\cos\gamma} dC \end{aligned} \tag{124}$$

- *Molin & Marion's formulation*

$$\{\boldsymbol{M}^{(2)}\} = \iint_{S_{B_0}} \left(P^{(2)} + P^{(1)}\left[\boldsymbol{\theta}^{(1)}\right]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS + \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(1)}\left[\boldsymbol{\theta}^{(1)}\right]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS$$

$$-\varrho g V z_B^{(0)} \left(\begin{Bmatrix} \theta_x^{(2)} \\ \theta_y^{(2)} \\ 0 \end{Bmatrix} - \frac{1}{2} \begin{Bmatrix} -\mathcal{H}_{23} \\ \mathcal{H}_{13} \\ 0 \end{Bmatrix} \right) + \frac{1}{2} \varrho g \int_{C_{B_0}} \left(\Xi^{(1)} - z^{(1)}\right)^2 \frac{[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}}{\cos\gamma} dC \tag{125}$$

It follows that, in order for the two formulations to be equivalent, the following identity should be verified:

$$I_{M_0} = -\varrho g \iint_{S_{B_0}} \left(z^{(2)} + z^{(0)}\left[\boldsymbol{A}^{(2)}\right] + z^{(1)}\left[\boldsymbol{\theta}^{(1)}\right]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS$$

$$= \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(1)}\left[\boldsymbol{\theta}^{(1)}\right]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS - \varrho g V z_B^{(0)} \left(\begin{Bmatrix} \theta_x^{(2)} \\ \theta_y^{(2)} \\ 0 \end{Bmatrix} - \frac{1}{2} \begin{Bmatrix} -\mathcal{H}_{23} \\ \mathcal{H}_{13} \\ 0 \end{Bmatrix} \right) \tag{126}$$

Similar to the forces, this identity follows directly from the divergence theorem but here we demonstrate it again using the inverse approach. For that purpose, the integral I_{M_0} is decomposed in two parts:

$$I_{M_0} = -\varrho g \iint_{S_{B_0} \pm S_{Fl_0}} \left(z^{(2)} + z^{(0)}\left[\boldsymbol{A}^{(2)}\right] + z^{(1)}\left[\boldsymbol{\theta}^{(1)}\right]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS = I_{M_0}^1 + I_{M_0}^2 \tag{127}$$

where $I_{M_0}^1$ is the integral over the closed volume and $I_{M_0}^2$ is the integral over the waterplane area.
Knowing that:

$$I_{M_0}^2 = \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(0)}\left[\boldsymbol{A}^{(2)}\right] + z^{(1)}\left[\boldsymbol{\theta}^{(1)}\right]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS = \varrho g \iint_{S_{Fl_0}} \left(z^{(2)} + z^{(1)}\left[\boldsymbol{\theta}^{(1)}\right]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS \tag{128}$$

it follows that, in order for (126) to be satisfied, the following should be true:

$$I_{M_0}^1 = -\varrho g \iint_{S_{B_0} + S_{Fl_0}} \left(z^{(2)} + z^{(0)}\left[\boldsymbol{A}^{(2)}\right] + z^{(1)}\left[\boldsymbol{\theta}^{(1)}\right]\right)[\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS = -\varrho g V z_B^{(0)} \left(\begin{Bmatrix} \theta_x^{(2)} \\ \theta_y^{(2)} \\ 0 \end{Bmatrix} - \frac{1}{2} \begin{Bmatrix} -\mathcal{H}_{23} \\ \mathcal{H}_{13} \\ 0 \end{Bmatrix} \right) \tag{129}$$

Each term in $I_{M_0}^1$ is developed separately:

$$\iint_{S_{B_0} + S_{Fl_0}} z^{(2)} [\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS = \iiint_V [\nabla]\left(z^{(2)}\{\boldsymbol{u}_0\}\right)dV = V z_B^{(0)} \left(\begin{Bmatrix} \theta_x^{(2)} \\ \theta_y^{(2)} \\ 0 \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} -\mathcal{H}_{32} \\ \mathcal{H}_{31} \\ 0 \end{Bmatrix} \right)$$

$$\iint_{S_{B_0} + S_{Fl_0}} z^{(0)} \left[\boldsymbol{A}^{(2)}\right] [\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS = \left(\left[\boldsymbol{\theta}^{(2)}\right] - \frac{1}{2}[\boldsymbol{\mathcal{H}}]\right) \iiint_V [\nabla]\left[\left(z_0 + z_G^{(0)}\right)\{\boldsymbol{u}_0\}\right]dV = 0$$

$$\iint_{S_{B_0} + S_{Fl_0}} z^{(1)} \left[\boldsymbol{A}^{(1)}\right] [\boldsymbol{u}_0]\{\boldsymbol{n}_0\}dS = \left[\boldsymbol{\theta}^{(1)}\right] \iiint_V [\nabla]\left[\left(z_G^{(1)} + \theta_x^{(1)} y_0 - \theta_y^{(1)} x_0\right)\{\boldsymbol{u}_0\}\right]dV = V z_B^{(0)} \begin{Bmatrix} -\theta_z^{(1)}\theta_y^{(1)} \\ \theta_z^{(1)}\theta_x^{(1)} \\ 0 \end{Bmatrix}$$

After summing the three terms it follows that:

$$V z_B^{(0)} \left(\begin{Bmatrix} \theta_x^{(2)} \\ \theta_y^{(2)} \\ 0 \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} -\mathcal{H}_{32} \\ \mathcal{H}_{31} \\ 0 \end{Bmatrix} \right) + V z_B^{(0)} \begin{Bmatrix} -\theta_z^{(1)}\theta_y^{(1)} \\ \theta_z^{(1)}\theta_x^{(1)} \\ 0 \end{Bmatrix} = V z_B^{(0)} \left(\begin{Bmatrix} \theta_x^{(2)} \\ \theta_y^{(2)} \\ 0 \end{Bmatrix} - \frac{1}{2} \begin{Bmatrix} -\mathcal{H}_{23} \\ \mathcal{H}_{13} \\ 0 \end{Bmatrix} \right) \tag{130}$$

Knowing that the elements of the matrix $[\boldsymbol{\mathcal{H}}]$ are given by (83) it can be concluded that the identity (130) is satisfied, which completes the proof of the equivalence of two formulations.

6.4 Ogilvie [4]

The following expression is given up to second order:

$$\begin{aligned}
\mathbf{G} = {} & \rho g V (y_b \mathbf{i} - x_b \mathbf{j}) - \varepsilon\rho \iint_{S_m} (\mathbf{x}\times\mathbf{n})\,\phi_{1_t}\, d\sigma \\
& - \varepsilon\rho g \begin{pmatrix} -V\xi_{21} + y_f A_{WP}\xi_{31} + (z_b V + L_{22})\xi_{41} - L_{12}\xi_{51} - x_b V\xi_{61} \\ V\xi_{11} - x_f A_{WP}\xi_{31} - L_{12}\xi_{41} + (z_b V + L_{11})\xi_{51} - y_b V\xi_{61} \\ 0 \end{pmatrix} \\
& - \varepsilon^2\rho \iint_{S_m} \Big((\mathbf{x}\times\mathbf{n})\big\{\phi_{2_t} + \tfrac{1}{2}|\nabla\phi_1|^2 + (\boldsymbol{\xi}_1 + \boldsymbol{\alpha}_1\times\mathbf{x})\cdot\nabla\phi_{1_t}\big\} \\
& \qquad\qquad + \big\{(\boldsymbol{\xi}_1\times\mathbf{n}) + \boldsymbol{\alpha}_1\times(\mathbf{x}\times\mathbf{n})\big\}\phi_{1_t}\Big)\, d\sigma \\
& + \frac{1}{2}\varepsilon^2\rho g \oint_{C_m} d\ell\, (\mathbf{x}\times\mathbf{n})\big\{\zeta_1^2 - 2\zeta_1[\xi_{31} + y\xi_{41} - x\xi_{51}]\big\} \\
& + \text{other second-order terms with } \mathbf{i} \text{ and } \mathbf{j} \text{ components} \\
& \qquad + O(\varepsilon^3)\,, \qquad\qquad (74)
\end{aligned}$$

Figure 8: Moments up to second order, by Ogilvie [4].

The reference point for the definition of the moments is the origin of the inertial coordinate system which is located at the mean position of the free surface with an arbitrary horizontal location. The expressions which are given in [4] are incomplete and do not allow for detailed comparisons.

It is also important to note that, in the Ogilvie's formulation, the convention "123" is used for the definition of the Euler angles (care when interpreting because the transpose of the transformation matrix is first introduced). This means that the antisymmetric matrix $[\boldsymbol{H}]_{AS}$ is:

$$[\boldsymbol{H}]_{AS} = \begin{bmatrix} 0 & -\theta_y^{(1)}\theta_x^{(1)} & -\theta_z^{(1)}\theta_x^{(1)} \\ \theta_y^{(1)}\theta_x^{(1)} & 0 & -\theta_z^{(1)}\theta_y^{(1)} \\ \theta_z^{(1)}\theta_x^{(1)} & \theta_z^{(1)}\theta_y^{(1)} & 0 \end{bmatrix} \tag{131}$$

This fact does not represent any particular problem, and the motions of the floating body should remain the same as for any other convention for defining the transformation matrix.

6.5 Pinkster [5]

The following expression is given for the second order external moment:

$$
\begin{aligned}
\overline{M}^{(2)} = & -\int_{WL} \tfrac{1}{2}\rho g \zeta_r^{(1)^2} . (\overline{x} \times \overline{n}) . d\ell + \overline{\alpha}^{(1)} \times (I . \ddot{\overline{\alpha}}^{(1)}) + \\
& -\iint_{S_0} \{-\tfrac{1}{2}\rho |\overline{\nabla}\Phi^{(1)}|^2 - \rho\Phi_t^{(2)} - \rho(\overline{X}^{(1)} . \overline{\nabla}\Phi_t^{(1)})\} . \\
& . (\overline{x} \times \overline{n}) . dS - \iint_{S_0} -\rho g X_3^{(2)} . (\overline{x} \times \overline{n}) . dS \\
& \qquad \ldots \ldots \ldots \ldots \quad \text{(III-74)}
\end{aligned}
$$

Figure 9: Second order moment by Pinkster.

It should be noted that the reference point for the definition of the moments is the instantaneous position of the center of gravity i.e. the same as in direct formulation and that by Molin & Marion [3]. It follows that the expression for the second order moment (Figure 9) is the same as the one given by direct formulation. Once again, the only difference is the fact that the quadratic term [$\mathcal{H}$] in the second order transformation matrix is missing.

7 Boundary Value Problems

The critical elements in the definition of the Boundary Value Problems for the velocity potentials at different order are the boundary conditions at the free surface and on the body.

7.1 Free surface boundary condition

The free surface boundary conditions are the same in all formulations. They are given by:

$$\frac{\partial^2\Phi^{(1)}}{\partial t^2} + g\frac{\partial\Phi^{(1)}}{\partial Z} = 0 \tag{132}$$

$$\frac{\partial^2\Phi^{(2)}}{\partial t^2} + g\frac{\partial\Phi^{(2)}}{\partial Z} = -2\nabla\Phi^{(1)}\nabla\frac{\partial\Phi^{(1)}}{\partial t} + \frac{1}{g}\frac{\partial\Phi^{(1)}}{\partial t}\left[\frac{\partial^3\Phi^{(1)}}{\partial t^2\partial Z} + g\frac{\partial^2\Phi^{(1)}}{\partial Z^2}\right] \tag{133}$$

The corresponding wave elevations are also the same and given by:

$$\Xi^{(1)} = -\frac{1}{g}\frac{\partial\Phi^{(1)}}{\partial t} \tag{134}$$

$$\Xi^{(2)} = -\frac{1}{g}\left[\frac{\partial\Phi^{(2)}}{\partial t} + \frac{1}{2}\left(\nabla\Phi^{(1)}\right)^2 - \frac{1}{g}\frac{\partial\Phi^{(1)}}{\partial t}\frac{\partial^2\Phi^{(1)}}{\partial t\partial Z}\right] \tag{135}$$

7.2 Body boundary condition

7.2.1 Direct formulation

The fluid velocity at the instantaneous position on the body is written as a function of its value at rest using the Taylor series expansion:

$$\nabla\Phi(\boldsymbol{r}) = \nabla\Phi(\boldsymbol{r_0}) + (\{\boldsymbol{r}-\boldsymbol{r_0}\}^T\{\nabla\})\nabla\Phi(\boldsymbol{r_0}) \tag{136}$$

The displacement of the point attached to the body being given by (23), we can write:

$$\{\nabla\Phi\} = \varepsilon\{\nabla\Phi^{(1)}\} + \varepsilon^2\left[\{\nabla\Phi^{(2)}\} + \left(\{\boldsymbol{r}^{(1)}\}^T\{\nabla\}\right)\{\nabla\Phi^{(1)}\}\right] \tag{137}$$

so that the body boundary conditions at first two orders become:

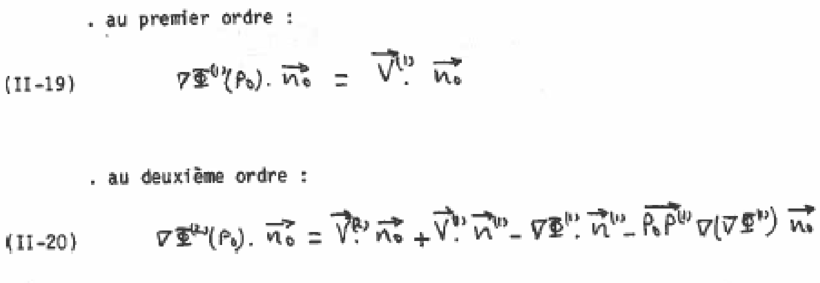
$$\{\nabla\Phi^{(1)}\}^T\{\boldsymbol{n}_0\} = \{\boldsymbol{v}^{(1)}\}^T\{\boldsymbol{n}_0\} \tag{138}$$

$$\{\nabla\Phi^{(2)}\}^T\{\boldsymbol{n}_0\} = \left[\{\boldsymbol{v}^{(2)}\} - \left(\{\boldsymbol{r}^{(1)}\}^T\{\nabla\}\right)\{\nabla\Phi^{(1)}\}\right]^T\{\boldsymbol{n}_0\} + \left(\{\boldsymbol{v}^{(1)}\} - \{\nabla\Phi^{(1)}\}\right)^T\{\boldsymbol{n}^{(1)}\} \tag{139}$$

where the relations (24) to (29) should be used for the different first and second order quantities.

7.2.2 Molin & Marion [3]

The following expressions are given:


. au premier ordre :

(II-19) $\nabla\Phi^{(1)}(P_0).\,\vec{n}_0 = \vec{V}^{(1)}.\,\vec{n}_0$

. au deuxième ordre :

(II-20) $\nabla\Phi^{(2)}(P_0).\,\vec{n}_0 = \vec{V}^{(2)}.\,\vec{n}_0 + \vec{V}^{(1)}.\,\vec{n}^{(1)} - \nabla\Phi^{(1)}.\,\vec{n}^{(1)} - \overrightarrow{P_0P^{(1)}}\,\nabla(\nabla\Phi^{(1)})\,\vec{n}_0$

Figure 10: Body boundary conditions at different orders, by Molin & Marion.

Within the present notations we have:

$$\vec{\boldsymbol{n}}_0 = \{\boldsymbol{n}_0\} \quad , \quad \overrightarrow{P_0P^{(1)}} = \{\boldsymbol{r}^{(1)}\} \quad , \quad \vec{\boldsymbol{n}}^{(1)} = \{\boldsymbol{n}^{(1)}\} \quad , \quad \vec{\boldsymbol{V}}^{(1)} = \{\boldsymbol{v}^{(1)}\} \quad , \quad \vec{\boldsymbol{V}}^{(2)} = \{\boldsymbol{v}^{(2)}\} \tag{140}$$

In Molin & Marion (1985), the velocity components $\vec{\boldsymbol{V}}^{(1)}$ and $\vec{\boldsymbol{V}}^{(2)}$ seem to not be given explicitly in time domain, but in frequency domain only. However they can be deduced by adding to the translational velocity of the center of gravity, the relative velocity of the point around the center of gravity which are given by:

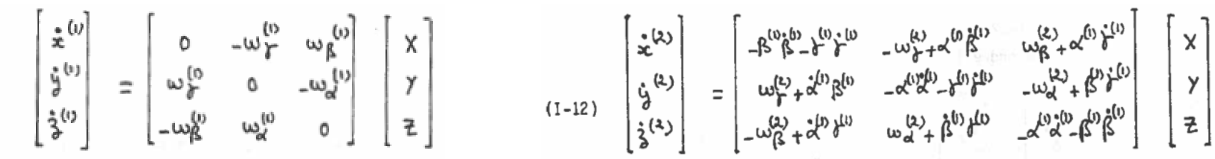


Figure 11: Velocity of the point attached to the body in Molin & Marion.

Using the notations from Molin & Marion, it follows that:

$$\vec{\boldsymbol{V}}^{(1)} = \begin{bmatrix} \dot{x}_G^{(1)} \\ \dot{y}_G^{(1)} \\ \dot{z}_G^{(1)} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\gamma}^{(1)} & \dot{\beta}^{(1)} \\ \dot{\gamma}^{(1)} & 0 & -\dot{\alpha}^{(1)} \\ -\dot{\beta}^{(1)} & \dot{\alpha}^{(1)} & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{141}$$

$$\begin{aligned} \vec{\boldsymbol{V}}^{(2)} = & \begin{bmatrix} \dot{x}_G^{(2)} \\ \dot{y}_G^{(2)} \\ \dot{z}_G^{(2)} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\gamma}^{(2)} & \dot{\beta}^{(2)} \\ \dot{\gamma}^{(2)} & 0 & -\dot{\alpha}^{(2)} \\ -\dot{\beta}^{(2)} & \dot{\alpha}^{(2)} & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \\ + & \begin{bmatrix} -\beta^{(1)}\dot{\beta}^{(1)} - \gamma^{(1)}\dot{\gamma}^{(1)} & \alpha^{(1)}\dot{\beta}^{(1)} & \alpha^{(1)}\dot{\gamma}^{(1)} \\ \dot{\alpha}^{(1)}\beta^{(1)} & -\alpha^{(1)}\dot{\alpha}^{(1)} - \gamma^{(1)}\dot{\gamma}^{(1)} & \beta^{(1)}\dot{\gamma}^{(1)} \\ \dot{\alpha}^{(1)}\gamma^{(1)} & \dot{\beta}^{(1)}\gamma^{(1)} & -\alpha^{(1)}\dot{\alpha}^{(1)} - \beta^{(1)}\dot{\beta}^{(1)} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \end{aligned} \tag{142}$$

Within the present notations $\vec{\boldsymbol{V}}^{(1)}$ and $\vec{\boldsymbol{V}}^{(2)}$ correspond to $\{\boldsymbol{v}^{(1)}\}$ and $\{\boldsymbol{v}^{(2)}\}$ which are recalled here in the form:

$$\{\boldsymbol{v}^{(1)}\} = \{\dot{\boldsymbol{r}}_G^{(1)}\} + [\dot{\boldsymbol{\theta}}^{(1)}]\{\boldsymbol{u}_0\} \quad , \quad \{\boldsymbol{v}^{(2)}\} = \{\dot{\boldsymbol{r}}_G^{(2)}\} + \left([\dot{\boldsymbol{\theta}}^{(2)}] - \frac{1}{2}[\dot{\boldsymbol{\mathcal{H}}}]\right)\{\boldsymbol{u}_0\} \tag{143}$$

Within the formulation of Molin & Marion, the quadratic part $[\boldsymbol{\mathcal{H}}]$, of the second order transformation matrix is given by $[\boldsymbol{\mathcal{H}}] = [\boldsymbol{H}]_S$ so that we have:

$$[\dot{\boldsymbol{\mathcal{H}}}] = [\dot{\boldsymbol{H}}]_S = \begin{bmatrix} 2\left(\theta_z^{(1)}\dot{\theta}_z^{(1)} + \theta_y^{(1)}\dot{\theta}_y^{(1)}\right) & -\dot{\theta}_y^{(1)}\theta_x^{(1)} - \theta_y^{(1)}\dot{\theta}_x^{(1)} & -\dot{\theta}_z^{(1)}\theta_x^{(1)} - \theta_z^{(1)}\dot{\theta}_x^{(1)} \\ -\dot{\theta}_y^{(1)}\theta_x^{(1)} - \theta_y^{(1)}\dot{\theta}_x^{(1)} & 2\left(\theta_z^{(1)}\dot{\theta}_z^{(1)} + \theta_x^{(1)}\dot{\theta}_x^{(1)}\right) & -\dot{\theta}_z^{(1)}\theta_y^{(1)} - \theta_z^{(1)}\dot{\theta}_y^{(1)} \\ -\dot{\theta}_z^{(1)}\theta_x^{(1)} - \theta_z^{(1)}\dot{\theta}_x^{(1)} & -\dot{\theta}_z^{(1)}\theta_y^{(1)} - \theta_z^{(1)}\dot{\theta}_y^{(1)} & 2\left(\theta_y^{(1)}\dot{\theta}_y^{(1)} + \theta_x^{(1)}\dot{\theta}_x^{(1)}\right) \end{bmatrix} \tag{144}$$

In Molin & Marion, the matrix $[\dot{\boldsymbol{\mathcal{H}}}]$ is further simplified by assuming the periodic motion, which apparently requires that:

$$[\dot{\boldsymbol{\theta}}^{(1)}][\boldsymbol{\theta}^{(1)}] = [\boldsymbol{\theta}^{(1)}][\dot{\boldsymbol{\theta}}^{(1)}] \tag{145}$$

so that finally we have:

$$-\frac{1}{2}[\dot{\boldsymbol{\mathcal{H}}}] = -\frac{1}{2}[\dot{\boldsymbol{H}}]_S = \begin{bmatrix} -\theta_z^{(1)}\dot{\theta}_z^{(1)} - \theta_y^{(1)}\dot{\theta}_y^{(1)} & \dot{\theta}_y^{(1)}\theta_x^{(1)} & \dot{\theta}_z^{(1)}\theta_x^{(1)} \\ \theta_y^{(1)}\dot{\theta}_x^{(1)} & -\theta_z^{(1)}\dot{\theta}_z^{(1)} - \theta_x^{(1)}\dot{\theta}_x^{(1)} & \dot{\theta}_z^{(1)}\theta_y^{(1)} \\ \theta_z^{(1)}\dot{\theta}_x^{(1)} & \theta_z^{(1)}\dot{\theta}_y^{(1)} & -\theta_y^{(1)}\dot{\theta}_y^{(1)} - \theta_x^{(1)}\dot{\theta}_x^{(1)} \end{bmatrix} \tag{146}$$

This is the last term in (142) so that the equivalence of the expressions (143) and (142) is demonstrated.
It is important to note that the quadratic term $[\boldsymbol{\mathcal{H}}]$ being different in the direct approach, within the "321" convention for Euler angles, the following expression is obtained for its time derivative:

$$-\frac{1}{2}[\dot{\boldsymbol{\mathcal{H}}}] = \begin{bmatrix} -\theta_z^{(1)}\dot{\theta}_z^{(1)} - \theta_y^{(1)}\dot{\theta}_y^{(1)} & \dot{\theta}_y^{(1)}\theta_x^{(1)} + \theta_y^{(1)}\dot{\theta}_x^{(1)} & \dot{\theta}_z^{(1)}\theta_x^{(1)} + \theta_z^{(1)}\dot{\theta}_x^{(1)} \\ 0 & -\theta_z^{(1)}\dot{\theta}_z^{(1)} - \theta_x^{(1)}\dot{\theta}_x^{(1)} & \dot{\theta}_z^{(1)}\theta_y^{(1)} + \theta_z^{(1)}\dot{\theta}_y^{(1)} \\ 0 & 0 & -\theta_y^{(1)}\dot{\theta}_y^{(1)} - \theta_x^{(1)}\dot{\theta}_x^{(1)} \end{bmatrix} \tag{147}$$

7.2.3 Ogilvie [4]

The expressions given by Ogilvie are:

$$n'\cdot\nabla\phi_1 = n'\cdot[\dot{\xi}_1 + \dot{\alpha}_1\times x'] \; ; \qquad (69a)$$

$$n'\cdot\nabla\phi_2 = n'\cdot\{[\dot{\xi}_2 + \dot{\alpha}_2\times x'] + \dot{H}x' - [(\xi_1 + \alpha_1\times x')\cdot\nabla]\nabla\phi_1\} + (\alpha_1\times n')\cdot[(\dot{\xi}_1 + \dot{\alpha}_1\times x') - \nabla\phi_1] \; . \qquad (69b)$$

Figure 12: Body boundary conditions at different orders, by Ogilvie

It can be seen that the formulation is the same as in the direct approach provided that the quadratic term $[\boldsymbol{\mathcal{H}}]$, in the second order transformation matrix, corresponds to the "123" convention for Euler angles.

7.2.4 Pinkster [5]

The expressions given by Pinkster are:

surface. The first order boundary condition becomes:

$$\overline{\nabla}\Phi^{(1)}.\overline{n} = \overline{V}^{(1)}.\overline{n} \quad \ldots\ldots\ldots\ldots \quad (III-33)$$

The second order boundary condition becomes:

$$\overline{\nabla}\Phi^{(2)}.\overline{n} = -(\overline{X}^{(1)}.\overline{\nabla}).\overline{\nabla}\Phi^{(1)}.\overline{n} + (\overline{V}^{(1)} - \overline{\nabla}\Phi^{(1)}).\overline{N}^{(1)} + + \overline{V}^{(2)}.\overline{n} \quad \ldots\ldots\ldots\ldots \quad (III-34)$$

Figure 13: Body boundary conditions at different orders, by Pinkster.

It can be seen that the expression takes the same form as in the direct approach. However, as already mentioned Pinkster formulation contains an error because the quadratic term $[\boldsymbol{\mathcal{H}}]$, in the second order transformation matrix, is missing, which is not correct.

7.2.5 Chen [1]

The expressions given by Chen are:

$$\frac{\partial}{\partial n}\Phi^{(K)}\Big|_{SCO} = \vec{V}_E^{(K)}.\vec{n}_o + A_C^{(K)}$$

$$A_C^{(1)} = 0$$

$$A_C^{(2)} = \left[\vec{V}_E^{(1)} - \nabla\Phi^{(1)}\right].R^{(1)}.\vec{n}_o - \left[(\overrightarrow{MOM}^{(1)}.\nabla).\nabla\Phi^{(1)}\right].\vec{n}_o\Big|_{SCO}$$

Figure 14: Body boundary conditions at different orders, by Chen.

where, using the present notations, the velocities $\vec{\boldsymbol{V}}_E^{(1)}$ and $\vec{\boldsymbol{V}}_E^{(2)}$ are defined by:

$$\vec{\boldsymbol{V}}_E^{(1)} = \left\{\dot{\boldsymbol{r}}_G^{(1)}\right\} + \left[\dot{\boldsymbol{\theta}}^{(1)}\right]\{\boldsymbol{u}_0\} \quad , \quad \vec{\boldsymbol{V}}_E^{(2)} = \left\{\dot{\boldsymbol{r}}_G^{(2)}\right\} + \left[\dot{\boldsymbol{\theta}}^{(2)}\right]\{\boldsymbol{u}_0\} \qquad (148)$$

This means that the quadratic term $[\dot{\boldsymbol{\mathcal{H}}}]$ is missing which is not correct.

8 Inertial loading & motion equations

8.1 General

The inertial loading follows from the conservation laws for the linear $\{\mathbf{P}\}$ and the angular momentum $\{\mathbf{L}\}$ which are defined as follows:

$$\{\mathbf{P}\} = [\boldsymbol{m}]\{\dot{\boldsymbol{r}}_G\} \quad , \quad \{\mathbf{L}\} = [\boldsymbol{I}_{\theta\theta}]\{\boldsymbol{\Omega}\} \tag{149}$$

where $[\boldsymbol{I}_{\theta\theta}]$ denotes the rotational inertia matrix which is defined by:

$$[\boldsymbol{I}_{\theta\theta}] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} = \iiint_{\forall_B} [\boldsymbol{u}]^T[\boldsymbol{u}]\,\rho dV = \iiint_{\forall_B} \begin{bmatrix} u_z^2 + u_y^2 & -u_y u_x & -u_z u_x \\ -u_x u_y & u_z^2 + u_x^2 & -u_z u_y \\ -u_x u_z & -u_y u_z & u_y^2 + u_x^2 \end{bmatrix} \rho dV \tag{150}$$

The corresponding time derivatives of the linear and the angular momentums (149) are given by:

$$\{\dot{\mathbf{P}}\} = [\boldsymbol{m}]\{\ddot{\boldsymbol{r}}_G\} \quad , \quad \{\dot{\mathbf{L}}\} = [\boldsymbol{I}_{\theta\theta}]\{\dot{\boldsymbol{\Omega}}\} + [\boldsymbol{\Omega}][\boldsymbol{I}_{\theta\theta}]\{\boldsymbol{\Omega}\} \tag{151}$$

Due to the body motions the rotational inertia matrix $[\boldsymbol{I}_{\theta\theta}]$ depends on time through the following relation:

$$[\boldsymbol{I}_{\theta\theta}] = [\boldsymbol{A}][\boldsymbol{I}'_{\theta\theta}][\boldsymbol{A}]^T \tag{152}$$

where $[\boldsymbol{I}'_{\theta\theta}]$ denotes the rotational inertia matrix expressed in the body fixed coordinate system:

$$[\boldsymbol{I}'_{\theta\theta}] = \iiint_{\forall_B} [\boldsymbol{u}_0]^T[\boldsymbol{u}_0]\,\rho dV = \iiint_{\forall_B} \begin{bmatrix} u_{0z}^2 + u_{0y}^2 & -u_{0y} u_{0x} & -u_{0z} u_{0x} \\ -u_{0x} u_{0y} & u_{0z}^2 + u_{0x}^2 & -u_{0z} u_{0y} \\ -u_{0x} u_{0z} & -u_{0y} u_{0z} & u_{0y}^2 + u_{0x}^2 \end{bmatrix} \rho dV \tag{153}$$

The second order rotational motion is considered in details only.

8.2 Direct approach

The relation (152) is first developed up to second order:

$$[\boldsymbol{I}_{\theta\theta}] = [\boldsymbol{I}'_{\theta\theta}] + \varepsilon\left([\boldsymbol{A}^{(1)}][\boldsymbol{I}'_{\theta\theta}] - [\boldsymbol{I}'_{\theta\theta}][\boldsymbol{A}^{(1)}]\right) \tag{154}$$

It follows that the time derivative of the angular momentum at second order becomes:

$$\{\dot{\mathbf{L}}^{(2)}\} = [\boldsymbol{I}'_{\theta\theta}]\{\dot{\boldsymbol{\Omega}}^{(2)}\} + \left([\boldsymbol{A}^{(1)}][\boldsymbol{I}'_{\theta\theta}] - [\boldsymbol{I}'_{\theta\theta}][\boldsymbol{A}^{(1)}]\right)\{\dot{\boldsymbol{\Omega}}^{(1)}\} + [\boldsymbol{\Omega}^{(1)}][\boldsymbol{I}'_{\theta\theta}]\{\boldsymbol{\Omega}^{(1)}\} \tag{155}$$

This can be rewritten in terms of the rotational angles as follows:

$$\{\dot{\mathbf{L}}^{(2)}\} = [\boldsymbol{I}'_{\theta\theta}]\left(\{\ddot{\boldsymbol{\theta}}^{(2)}\} + [\dot{\mathbf{G}}^{(1)}]\{\dot{\boldsymbol{\theta}}^{(1)}\} + [\mathbf{G}^{(1)}]\{\ddot{\boldsymbol{\theta}}^{(1)}\}\right) + \left([\boldsymbol{\theta}^{(1)}][\boldsymbol{I}'_{\theta\theta}] - [\boldsymbol{I}'_{\theta\theta}][\boldsymbol{\theta}^{(1)}]\right)\{\ddot{\boldsymbol{\theta}}^{(1)}\} + [\dot{\boldsymbol{\theta}}^{(1)}][\boldsymbol{I}'_{\theta\theta}]\{\dot{\boldsymbol{\theta}}^{(1)}\} \tag{156}$$

In frequency domain this becomes:

$$\{\dot{\mathbf{L}}^{(2)}\} = -4\omega^2[\boldsymbol{I}'_{\theta\theta}]\{\boldsymbol{\theta}^{(2)}\} - \omega^2\left([\boldsymbol{I}'_{\theta\theta}][\mathbf{G}^{(1)}]\{\boldsymbol{\theta}^{(1)}\} + [\boldsymbol{\theta}^{(1)}][\boldsymbol{I}'_{\theta\theta}]\{\boldsymbol{\theta}^{(1)}\}\right) \tag{157}$$

8.3 Molin & Marion [3]

It can be shown that, in this case we have:

$$[\mathbf{G}^{(1)}] = \frac{1}{2}[\boldsymbol{\theta}^{(1)}] \tag{158}$$

so that we can write:

$$\{\dot{\mathbf{L}}^{(2)}\} = [\boldsymbol{I}'_{\theta\theta}]\{\ddot{\boldsymbol{\theta}}^{(2)}\} + \left([\boldsymbol{\theta}^{(1)}][\boldsymbol{I}'_{\theta\theta}] - \frac{1}{2}[\boldsymbol{I}'_{\theta\theta}][\boldsymbol{\theta}^{(1)}]\right)\{\ddot{\boldsymbol{\theta}}^{(1)}\} + [\dot{\boldsymbol{\theta}}^{(1)}][\boldsymbol{I}'_{\theta\theta}]\{\dot{\boldsymbol{\theta}}^{(1)}\} \tag{159}$$

In frequency domain this becomes:

$$\{\dot{\mathbf{L}}^{(2)}\} = -4\omega^2[\boldsymbol{I}'_{\theta\theta}]\{\boldsymbol{\theta}^{(2)}\} - \omega^2[\boldsymbol{\theta}^{(1)}][\boldsymbol{I}'_{\theta\theta}]\{\boldsymbol{\theta}^{(1)}\} \tag{160}$$

In [3], the final equilibrium equation for the rotational motion is given in frequency domain as:

$$\text{(I-25)} \quad -4\omega^2 I \begin{bmatrix} \alpha_2^{(2)} \\ \beta_2^{(2)} \\ \gamma_2^{(2)} \end{bmatrix} = \begin{bmatrix} C_{\alpha 2}^{(2)} \\ C_{\beta 2}^{(2)} \\ C_{\gamma 2}^{(2)} \end{bmatrix} + \omega^2 \begin{bmatrix} \beta_1^{(1)}\gamma_1^{(1)}(I_{ZZ}-I_{YY}) + \alpha_1^{(1)}\beta_1^{(1)} I_{XZ} \\ \alpha_1^{(1)}\gamma_1^{(1)}(I_{XX}-I_{ZZ}) + (\gamma_1^{(1)2}-\alpha_1^{(1)2}) I_{XZ} \\ \alpha_1^{(1)}\beta_1^{(1)}(I_{YY}-I_{XX}) + \beta_1^{(1)}\gamma_1^{(1)} I_{XZ} \end{bmatrix} \tag{161}$$

where the body was assumed to be symmetric i.e. $I_{XY} = I_{YZ} = 0$.
In order for (160) and (161) to be equivalent, the following identity should be verified:

$$[\boldsymbol{\theta}^{(1)}][\boldsymbol{I}'_{\theta\theta}]\{\boldsymbol{\theta}^{(1)}\} = \begin{Bmatrix} \theta_y^{(1)}\theta_z^{(1)}(I'_{zz} - I'_{yy}) + \theta_x^{(1)}\theta_y^{(1)}I'_{xz} \\ \theta_x^{(1)}\theta_z^{(1)}(I'_{xx} - I'_{zz}) + \left(\theta_z^{(1)^2} - \theta_x^{(1)^2}\right)I'_{xz} \\ \theta_x^{(1)}\theta_y^{(1)}(I'_{yy} - I'_{xx}) - \theta_y^{(1)}\theta_z^{(1)}I'_{xz} \end{Bmatrix} \tag{162}$$

This can be easily demonstrated by developing $[\boldsymbol{\theta}^{(1)}][\boldsymbol{I}'_{\theta\theta}]\{\boldsymbol{\theta}^{(1)}\}$. When doing that a typo error was found in the expression (161). The sign of last term in the quadratic inertia vector should change $\beta^{(1)}\gamma^{(1)}\mathrm{I}_{XZ} \Rightarrow -\beta^{(1)}\gamma^{(1)}\mathrm{I}_{XZ}$.

8.4 Ogilvie [4]

In [4] the description of the body motion is very particular and quite different from others.

8.4.1 Translation

The reference translation vector, for the definition of the body motion, is the translation of the origin of the body coordinate system and not the translation of the center of gravity, like in other formulations. The origin of the mean body fixed coordinate system (equal to the earth fixed coordinate system at initial instant) is placed at an arbitrary point in the plane of the free surface. This means that the unknowns of the translational (force) component of the body motion equation are the components of this reference translation and the translation of the body center of gravity is deduced in the second step, after solving the complete motion equation. The reference translation vector (motion of the origin of the coordinate system) is denoted by:

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \tag{163}$$

The body motion equation is deduced from the conservation of the linear and angular momentum (Euler Newton law) where the reference translational motion is the translation of the center of gravity:

$$\mathbf{F} = m\ddot{\mathbf{X}}_G \tag{164}$$

This particular description of the body dynamics leads to quite unusual expressions for the translational component of the body motion equation, in terms of the unknown translational vector :

$$\begin{aligned} m[\ddot{\boldsymbol{\xi}}_1 + \ddot{\boldsymbol{\alpha}}_1 \times \mathbf{X}'_G] &= \mathbf{F}_1 \\ m[\ddot{\boldsymbol{\xi}}_2 + \ddot{\boldsymbol{\alpha}}_2 \times \mathbf{X}'_G] &= \mathbf{F}_2 - m\ddot{H}\mathbf{X}'_G \end{aligned} \tag{165}$$

8.4.2 Rotation

The rotational component of the body motion equation is written as:

$$\mathbf{G}_G = \frac{d\mathbf{K}}{dt} \tag{166}$$

where, within the present notations, $\mathbf{G}_\mathrm{G}$ corresponds to $\{\boldsymbol{M}\}$ and $\mathbf{K}$ corresponds to $\{\mathbf{L}\}$.
The instantaneous orientation of the body (rotation) is described using the concept of the Euler angles with the "123" convention. The final rotational part of the body motion equation is given in the following form:

$$I\dot{\omega}'_1 = G_1 - x'_G \times F_1 \;; \qquad (82a)$$
$$I\dot{\omega}'_2 = G_2 - x'_G\times F_2 - \omega'_1\times I\omega'_1 - \alpha_1\times G_1 - \xi_1\times F_1 + x'_G\times(\alpha_1\times F_1) \;. \qquad (82b) \tag{167}$$

The different terms on the right hand side of (167) occur because of the particular choice of the description of the body motion which requires the transfer of the moments from the origin of the inertial coordinate system to the instantaneous center of gravity and, in addition, the transfer of the different quantities from the inertial to the body fixed coordinate system.
As it can be seen the motion equation (167) is written in terms of the rotational velocity vector ω' which, in addition, is expressed relative to the body fixed coordinate system. Within the present notations, this vector corresponds to:

$$\omega' = [\boldsymbol{A}]^T\{\boldsymbol{\Omega}\} = [\boldsymbol{A}]^T[\mathbf{G}]\{\dot{\boldsymbol{\theta}}\} \tag{168}$$

which gives at different orders:

$$\omega'_1 = \{\boldsymbol{\Omega}'^{(1)}\} = \{\dot{\boldsymbol{\theta}}^{(1)}\} \quad , \quad \omega'_2 = \{\boldsymbol{\Omega}'^{(2)}\} = \{\dot{\boldsymbol{\theta}}^{(2)}\} + [\mathbf{G}'^{(1)}]\{\dot{\boldsymbol{\theta}}^{(1)}\} \tag{169}$$

$$\dot{\omega}'_1 = \{\dot{\boldsymbol{\Omega}}'^{(1)}\} = \{\ddot{\boldsymbol{\theta}}^{(1)}\} \quad , \quad \dot{\omega}'_2 = \{\dot{\boldsymbol{\Omega}}'^{(2)}\} = \{\ddot{\boldsymbol{\theta}}^{(2)}\} + [\dot{\mathbf{G}}'^{(1)}]\{\dot{\boldsymbol{\theta}}^{(1)}\} + [\mathbf{G}'^{(1)}]\{\ddot{\boldsymbol{\theta}}^{(1)}\} \tag{170}$$

In principle, there is nothing wrong with the motion equation (167) but it looks unnecessary complicated compared to the equivalent expression which could be obtained from the direct approach in the following form:

$$[\boldsymbol{I}'_{\theta\theta}]\{\dot{\boldsymbol{\Omega}}'^{(2)}\} = \{\boldsymbol{M}'^{(2)}\} - [\boldsymbol{\Omega}'^{(1)}][\boldsymbol{I}'_{\theta\theta}]\{\boldsymbol{\Omega}'^{(1)}\} \tag{171}$$

In addition, the fact that the motion equation is written with respect to the rotational velocity vector requires the additional step when evaluating the instantaneous orientation of the body (rotation angles $\{\boldsymbol{\theta}\}$), at each time step.

8.5 Pinkster [5]

The second order motion equation is not discussed, at least not in [5].

8.6 Chen [1]

In Chen [1] the moment of inertia is defined as:

$$(4.14) \quad \vec{M}(t) = \iiint_{V} \vec{r}_o \wedge \vec{a}(M,t)\,dm = \rho\frac{d^2}{dt^2}\iiint_{V} \vec{r}_o \wedge \overrightarrow{MoM}(t)\,dV \tag{172}$$

It looks like the expression is not correct because the acceleration is defined in the earth fixed (inertial) coordinate system while the position vector is defined in the body fixed coordinate system. The correct expression is:

$$\vec{M} = \iiint_{\forall} \vec{r} \wedge \vec{a}\rho dV \tag{173}$$

where $\vec{r}$ is the position vector relative to the instantaneous position of the center of gravity, expressed in the earth fixed coordinate system.
Using the notations from [1], the vectors $\vec{r}$ and $\vec{r}_0$ are related through the transformation matrix $R(\square)$ as follows:

$$\vec{r} = R(\vec{r}_0) \tag{174}$$

We can now rewrite the expression (173) as:

$$\vec{M} = \iiint_{\forall} \vec{r}_0 \wedge \vec{a}\rho dV + \iiint_{\forall} R(\vec{r}_0) \wedge \vec{a}\rho dV \tag{175}$$

which means that the second term is missing in the expression (172).
The expression (173) can also be rewritten as:

$$\vec{M} = \frac{d}{dt}\iiint_{\forall} \vec{r} \wedge \vec{v}\rho dV \tag{176}$$

In order to demonstrate that this expression is valid, we recall the following definitions:

$$
\begin{aligned}
\frac{d}{dt}\vec{r} &= \vec{\Omega} \wedge \vec{r} \\
\vec{v} &= \vec{v}_G + \vec{\Omega} \wedge \vec{r} \\
\vec{a} &= \frac{d}{dt}\vec{v} = \vec{a}_G + \dot{\vec{\Omega}} \wedge \vec{r} + \vec{\Omega} \wedge \left(\vec{\Omega} \wedge \vec{r}\right)
\end{aligned} \tag{177}
$$

Now we develop the expression (176) as follows:

$$
\vec{M} = \iiint_{\forall} \left(\frac{d}{dt}\vec{r} \wedge \vec{v} + \vec{r} \wedge \frac{d}{dt}\vec{v}\right)\rho dV \tag{178}
$$

It follows that in order for (176) to be valid, the following should be true:

$$
\vec{M} = \iiint_{\forall} \frac{d}{dt}\vec{r} \wedge \vec{v}\rho dV = 0 \tag{179}
$$

This can be easily demonstrated by using the following identities:

$$
\iiint_{\forall} \vec{r}\rho dV = 0 \quad , \quad \iiint_{\forall} \left(\vec{\Omega} \wedge \vec{r}\right) \wedge \vec{v}\rho dV = 0 \tag{180}
$$

Finally, within the present notations we can write:

$$
\vec{M} = \{\dot{\mathbf{L}}\} = [\boldsymbol{I}_{\theta\theta}]\{\dot{\boldsymbol{\Omega}}\} + [\boldsymbol{\Omega}][\boldsymbol{I}_{\theta\theta}]\{\boldsymbol{\Omega}\} \tag{181}
$$

This proves that the expression (173) is fully equivalent to the expression (151), which was obtained by considering the conservation of the angular momentum.

9 Discussions

Here we summarize the main characteristics of the different formulations and we compare them to the direct approach which was proposed here.

9.1 Molin & Marion [3]

- The description of the body motion is bit particular when compared to other formulations. This means that the transformation matrix is not explicitly defined for the fully nonlinear body motions. However, up to second order the same expressions as those given by Chen [1] are obtained. The particularity of this description of the body motion is that the antisymmetric part of the quadratic term, in the definition of the second order transformation matrix $\left[\boldsymbol{A}^{(2)}\right]$, is zero:

$$[\boldsymbol{H}]_{AS} = [0] \tag{182}$$

- The symmetric part of the second order transformation matrix is the same as in other formulations, and is given by:

$$[\boldsymbol{H}]_{S} = -\left[\boldsymbol{\theta}^{(1)}\right]\left[\boldsymbol{\theta}^{(1)}\right] \tag{183}$$

- The body motion equation in the time domain is not explicitly given (at different orders) but in the frequency domain only.
- In this formulation everything agrees with the direct formulation except the sign of the last term in the rotational inertia moment (161) which should be negative. This means that the following change should be made:

$$\beta^{(1)}\gamma^{(1)}\mathrm{I}_{XZ} \Rightarrow -\beta^{(1)}\gamma^{(1)}\mathrm{I}_{XZ} \tag{184}$$

9.2 Ogilvie [4]

- The description of the body dynamics is different from other formulations in the sense that the reference translation is defined as the translation of the origin of the coordinate system fixed to the body, which is not necessarily the body center of gravity. This fact give rise to some additional components in the excitation loading which occur when formulating the body motion equation.
- The concept of Euler angles is used to describe the nonlinear orientation of the body, within the so called "123" convention. Care should be taken when interpreting the transformation matrix because its transpose is first introduced.
- The external pressure forces agree with the direct formulation for the wall-sided bodies. The case of non wall-sided bodies was not considered.
- The external pressure moments are defined with respect to the origin of the inertial coordinate system, which was chosen to be located at the free surface with an arbitrary horizontal position. The expressions given for the moments appears to be incomplete and it was not possible to make the direct comparisons. Having said that, if the developments of the external moments have been performed till the end, most probably the same results would be obtained, because the same basic principles for their derivation are used.
- The Boundary Value Problems for the velocity potentials agree with the direct approach.

9.3 Pinkster [5]

- The body motion equation is not considered but only the external loads and the BVP's for the velocity potentials.
- The fully nonlinear description of the body motion was not considered and the development at different orders is given directly, which led to the fact that the quadratic term in the second order transformation matrix $[\boldsymbol{\mathcal{H}}]$ has been forgotten. This represents the main drawback of the Pinksters formulation and it has the consequences on several issues (external forces, body boundary condition, motion equations …)
- When considering the external pressure loading, the wall-sided bodies are considered only.
- Apart from the missing term $[\boldsymbol{\mathcal{H}}]$, the expressions for the external pressure loading and the formulations of the BVP's agree with the direct approach.

9.4 Chen [1]

- The body motion is described using the Rodriguez formula (e.g. see Shabana [6]). Up to second order, this description leads to the same expressions as those given by Molin & Marion [3].
- The pressure loading is first evaluated with respect to the initial position of the origin of the inertial (earth fixed) coordinate system, and is transferred to the instantaneous position of the center of gravity before solving the motion equation.
- The body motion equation is given both in the frequency and in the time domain.
- All the relevant aspects of the second order formulation agree with the direct approach except:
  - The term $[\dot{\mathcal{H}}]$ is missing in the definition of the body boundary condition for the second order velocity potential.
  - In the definition of the rotational component of the body inertia loading, the following term is missing:

$$\iiint_{\forall} R(\vec{r}_0) \wedge \vec{a} \rho dV \tag{185}$$

10 Conclusions

A comprehensive review of the most common formulations for the interactions of water waves and the floating rigid body is presented. The different notations, coordinate systems, description of nonlinear body motion …, made the exercise quite complex. It has been shown that the different formulations are fully equivalent in principle, except that some small errors are present in some of the formulations.

Traditionally the second order problem is usually formulated with respect to the earth fixed coordinate system and that is why the direct formulation which has been proposed here is also formulated in the earth fixed coordinate system. The formulation in the body fixed coordinate system was not discussed. Formulating the problem in the body fixed coordinate system leads, of course, to exactly the same results, however the components of the different vector quantities change. The interest for the formulation in the body fixed coordinate system is driven by the possibility to easily extend the theory to flexible bodies where the additional flexible modes are naturally defined with respect to the body fixed coordinate system. This work will be presented separately.

11 References

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