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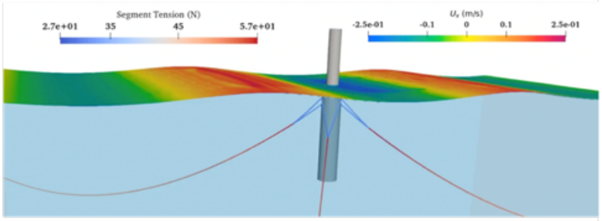

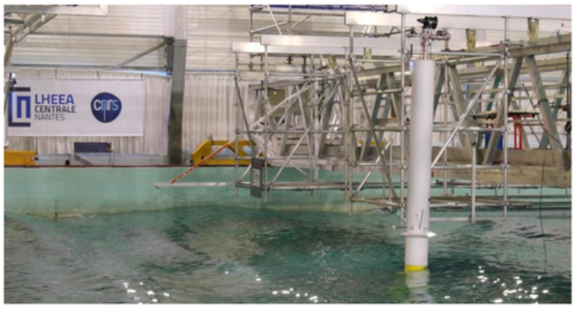

# Contrôle de la propagation des vagues par des barrières oscillantes verticalement

## *Controlling water waves by time-varying vertical plates*

**M. Koukouraki[1*], P. Petitjeans[1], A. Maurel[2] and V. Pagneux[3]**
**magdalini.koukouraki@espci.fr*

[1]Laboratoire de Physique et Mécanique des Milieux Hétérogènes, ESPCI-PSL, Paris, France
[2]Institut Langevin, ESPCI-PSL, Paris, France
[3]Laboratoire d'Acoustique de l' Université du Mans, Le Mans, France

**Résumé**

On s'intéresse à la propagation des ondes à la surface de l'eau dans une topographie variable en temps consistant en des plaques verticales infiniment minces qui oscillent au cours du temps. On commence par le problème de réflexion et de transmission d'une onde plane par une plaque verticale immergée dans un canal et on retrouve les coefficients de scattering en fonction de la fréquence de l'onde incidente et de la hauteur de la plaque. On présente une étude théorique basée sur le théorème de Floquet pour résoudre le problème lorsque la plaque oscille verticalement dans l'approximation basse fréquence et on observe la génération des harmoniques. On définit aussi la limite quasistatique pour une oscillation suffisamment lente de la barrière, une approximation qui apparaît assez robuste. La generation des harmoniques est aussi observée expérimentalement dans le cas d'une profondeur d'eau intermédiaire. Enfin, en utilisant un réseau périodique de plaques on étudie théoriquement, numériquement et expérimentalement le problème de déviation d'un paquet d'ondes, en passant d'un milieux isotrope à anisotrope à un instant donné.

**Summary**

This study focuses on the propagation of water waves over a time-varying topography, which consists of thin vertical plates as metamaterials. Starting with the scattering problem of a plane wave incident on a vertical submerged plate inside a water channel, we extract the scattering coefficients as a function of the frequency and the plate height. We then implement a Floquet theory approach for the scattering of the monochromatic wave by a vertically oscillating plate in the shallow water approximation, and show that $n$-harmonics are generated around the fundamental frequency. For a slowly oscillating plate we propose a quasistatic approximation which is in fact quite robust. The effect of harmonic generation is also observed experimentally in the intermediate depth regime. Finally, we extend the study to the case of a periodic array of plates sitting on the fluid

bottom in a two-dimensional space and investigate analytically, numerically and experimentally the propagation of shallow water wavepackets over a time-varying medium, which switches from isotropic to anisotropic at a given time.

# I – Introduction

Time-varying systems that involve wave-matter interactions have been widely explored over the years, as they reveal fascinating wave phenomena that are applicable across various fields, from quantum mechanics to condensed matter physics and fluid dynamics. These systems have introduced innovative methods for controlling and harnessing waves through effects like time reversal [1], frequency conversion [2], parametric amplification [3], temporal waveguiding [4]. Ever since Morgenthaler's groundbreaking work in the 1950s [5] on waves traveling through media with rapidly changing phase velocity, the research on time-varying metamaterials as a method to manipulate waves has expanded considerably [6]. Comprehending the interaction of waves with time-varying scatterers can be complex and frequently necessitates the creation of new theoretical and numerical approaches. For instance, Floquet theory has been long applied for the scattering of waves by periodically driven systems, as discussed in [7] and [8]. In the field of water waves, the topography can significantly influence the wave dynamics and some studies have already been made in that context, more specifically on moving underwater barriers and time-varying topographies (see [11] and [12]). In this work, we concentrate on vertical rigid plates with a small width, capable of undergoing prescribed vertical motion, and aim to explore various methods of controlling water waves. By using these plates as metamaterials we uncover wave phenomena such as harmonic generation (in the case of a single plate) and wave deflection (for a plate array).

# II – Single Oscillating Plate

## II – 1 The static plate for any water depth regime

### II – 1.1 Governing Equations

We consider an incompressible, irrotational and inviscid fluid of depth $h^+$, which extends horizontally along the $x$ axis in an unbounded domain, and an infinitely thin plate of height $h_p$ standing at the fluid bottom at location $x = 0$ (see Fig. 1). We wish to study the scattering problem of a plane wave arriving at the plate from $x = -\infty$, and retrieve the scattering coefficients for any given dimensionless frequency of the incident wave $\omega\sqrt{h^+/g}$. In the classical linearised water-wave theory (introduced in the textbooks [13]-[16]) and for a time-harmonic regime (convention $e^{-i\omega t}$), this problem reads as :

$$\begin{cases} \Delta\Phi = 0, \quad \text{in} \quad \Omega \\ \hat{\mathrm{n}} \cdot \nabla\Phi = 0, \quad \text{on} \quad \Gamma \\ \frac{\partial\Phi}{\partial z} = \frac{\omega^2}{g}\Phi, \quad z = 0, \end{cases} \tag{1}$$

where $\Phi(x, z, t)$ stands for the velocity potential, $\omega$ is the frequency of the monochromatic wave, $g$ the acceleration of gravity and $\hat{\mathrm{n}}$ the unit normal vector on $\Gamma$. The system (1) can be solved via mode matching method in order to recover the two-dimensional field for any given frequency and dimensionless plate height $\mu = h_p/h^+$. In Fig. 2a we show an example of the wave-field profile for $\omega\sqrt{h^+/g} = 1$, $\mu = 0.5$ and in Fig. 2b we depict the variation of the scattering coefficients with $\omega\sqrt{h^+/g}$ for two cases of plate heights. Notice that reflection is the strongest in the intermediate depth regime and that by increasing the plate height we amplify the reflection of the incident wave.

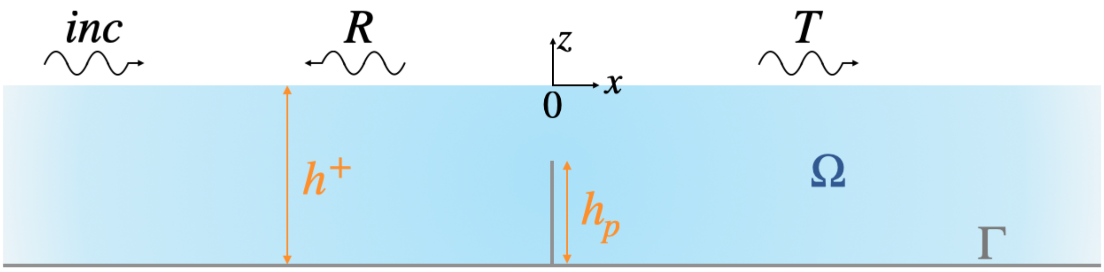


FIGURE 1 – Schematic representation of wave scattering by an infinitely thin plate inside a channel.

### II – 1.2 Shallow water model with jump conditions

In the long-wavelength limit, i.e. when the typical wavelength is much larger than the water depth, the wave-field becomes quasi-homogeneous along $z$, which allows for the simplification of the system (1) into one partial differential equation for the field at the surface. As it has been already addressed by Tuck [12], by starting from the system (1) and applying matched asymptotic expansions, one can obtain a reduced one-dimensional model where the effect of the plate is encapsulated with jump conditions at $x = 0$. This model is expressed as

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \tag{2a}$$

$$[\phi]_{0^-}^{0^+} = 2B_\mu h \partial_x \phi|_0 \,, \quad [\partial_x \phi]_{0^-}^{0^+} = 0, \tag{2b}$$

where $\phi(x,t)$ denotes now the $z$-independent velocity potential, $c_0 = \sqrt{gh}$ is the velocity of shallow water waves and $B_\mu$ is known as the blockage coefficient. $B_\mu$ can be determined in closed form for a range of fluid bottom profiles (as discussed in [9]) and for an infinitely thin plate is defined as

$$B_\mu = -\frac{2}{\pi}\ln\left[\sin\left(\frac{\pi}{2}(1-\mu)\right)\right]. \tag{3}$$

By considering once again the harmonic regime, we can write that $\phi = \Re\{f(x)e^{-i\omega t}\}$. For $x < 0$, the solution is simply given by the sum of the incident wave and the reflected wave, $f^- = e^{ikx} + R_{sw}e^{-ikx}$, while for $x > 0$ the solution consists of a transmitted wave, $f^+ = T_{sw}e^{ikx}$, with $k$ the wavenumber. By using these relations when evaluating the boundary conditions at $x = 0$ (Eq. (2b)), one finds that in the shallow water approximation

$$R_{sw} = -\frac{ikhB_\mu}{1 - ikhB_\mu}, \quad T_{sw} = \frac{1}{1 - ikhB_\mu}. \tag{4}$$

As illustrated in Fig. 2b, the approximation given by Eq. (4) (which is shown with the dotted red line) can efficiently describe the wave dynamics up to approximately $\omega\sqrt{h^+/g} = 0.2$, which gives us the limit of this shallow water regime.

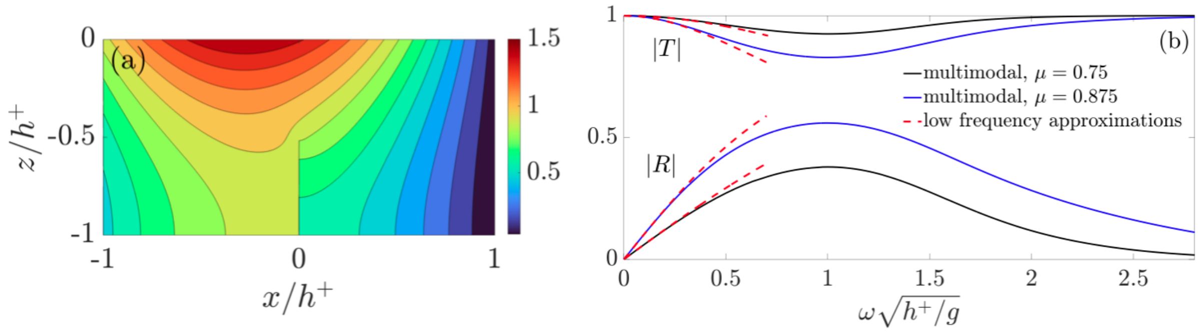


FIGURE 2 – (a) Wave field for $\omega\sqrt{h^+/g} = 1$, $\mu = 0.5$. (b) $|T|$ and $|R|$ with respect to $\omega\sqrt{h^+/g}$ for $\mu = 0.875, 0.75$ and the low frequency approximations given by Eq. (4).

## II – 2 The vertically oscillating plate

### II – 2.1 Floquet scattering in the shallow water regime

In this part we are interested in the time-varying case where $h_p = h_p(t)$ and wish to understand the interaction of the monochromatic wave with the moving barrier in the shallow water limit. For this purpose, we choose to work with the simplest case where the time dependence is incorporated in the blockage coefficient, such that

$$B_\mu(t) = B_{\mu,0} + B_{\mu,1}\cos(\omega_p t), \tag{5}$$

with $\omega_p$ the characteristic parameter of oscillation. Consequently, this leads to a more complex plate motion, defined as

$$\mu(t) = 1 - \frac{2}{\pi}\,\mathrm{asin}\left[\exp\left(-\frac{\pi}{2}B_\mu(t)\right)\right]. \tag{6}$$

Our starting point is the wave equation (2a) supplemented with the boundary conditions (2b), where $B_\mu$ follows Eq. (5). Note that since the plate is assumed to be infinitely thin, it does not act as a source when oscillating, which means that the reduced model (2b) also holds for the time-varying case with now a time-dependent discontinuity condition for $\phi$. Hence, in order to solve this problem, we implement the Floquet theorem and write the solution as

$$\phi = \Re\{e^{-i\omega t}\psi\}, \quad \psi(x,t) = \psi(x, t+T_p), \tag{7}$$

with $T_p = 2\pi/\omega_p$. Since $\psi$ is periodic, we can Fourier expand it as

$$\psi(x,t) = \sum_n \psi_n(x) e^{-in\omega_p t}, \tag{8}$$

with $n \in Z$ and the Fourier modes $e^{-in\omega_p t}$ satisfying the orthogonality relation :

$$\frac{1}{T_p}\int_0^{T_p} e^{i(m-n)\omega_p t}dt = \delta_{mn}. \tag{9}$$

Substituting Eqs. (7) and (8) into Eq. (2a), projecting on $e^{-in\omega_p t}$ and utilizing Eq. (9), we obtain the relation

$$\frac{d^2\psi_n}{dx^2} + k_n^2\psi_n = 0, \quad k_n = \frac{\omega_n}{c_0}, \tag{10}$$

where $\omega_n = \omega + n\omega_p$. Thus, we can express the solution for $\psi_n$ in the regions $x < 0$ and $x > 0$ as follows :

$$\psi_n^-(x<0) = \delta_{0n} e^{ik_n x} + r_n e^{-ik_n x}, \quad \psi_n^+(x>0) = t_n e^{ik_n x}, \tag{11}$$

with $r_n$ and $t_n$ the scattering coefficients of each harmonic. We proceed then to evaluate the jump conditions by plugging relations (11) in Eq. (2b), in order to derive the expressions for $r_n$ and $t_n$. By doing so, we find that

$$\vec{t} = \vec{b} - \vec{r}, \quad \vec{r} = -(2\mathrm{I} - \mathrm{V})^{-1}\mathrm{V}\vec{b}, \tag{12}$$

with

$$(\mathrm{V})_{m,m'} = iB_{\mu,1}h\left(k_{m'+1}\delta_{m,m'+1} + k_{m'-1}\delta_{m,m'-1}\right) + 2iB_{\mu,0}hk_m\delta_{m,m'}, \tag{13}$$

$(b)_m = \delta_{0m}$ and I denoting the identity matrix. Notice that $r_m = -t_m$ for $m \neq 0$, which arises from the symmetry of the problem, and that there is a coupling between the harmonics, meaning that the incident wave is scattered into $n$-sidebands with frequencies $\omega_n$.

Besides this Floquet formalism of the problem, which describes the full system for any given value of $\omega_p/\omega$ as long as the shallow water approximation is valid, it is worth exploring a limiting case where the plate oscillates much slower than the period of the incident wave, i.e. when $\omega_p/\omega \ll 1$. In this quasistatic adiabatic approximation, the solution for the reflected and the transmitted waves can be constructed by considering the static solution with the scattering coefficients of Eq. (4), with the modified blockage coefficient of Eq. (5). Hence, we write that

$$f_{r,QS} = -\frac{ikh(B_{\mu,0} + B_{\mu,1}\cos(\omega_p t))}{1 - ikh(B_{\mu,0} + B_{\mu,1}\cos(\omega_p t))} e^{-i(kx+\omega t)}, \tag{14}$$

$$f_{t,QS} = \frac{1}{1 - ikh(B_{\mu,0} + B_{\mu,1}\cos(\omega_p t))} e^{i(kx-\omega t)}, \tag{15}$$

with $f_{r,QS}$ and $f_{t,QS}$ denoting the reflected and transmitted waves in this quasistatic (QS) limit. Then, we expand the two components in a Fourier series :

$$f_{r,QS} = \sum_n \tilde{r}_n e^{i(kx-\omega_n t)}, \quad f_{t,QS} = \sum_n \tilde{t}_n e^{i(kx-\omega_n t)}, \tag{16}$$

with $\omega_n = \omega + n\omega_p$. Combining Eqs. (14), (15) with the relations (16), then projecting on $e^{-in\omega_p t}$ and using the orthogonality of the modes, one can express the coefficients $\tilde{r}_n$ and $\tilde{t}_n$ as

$$\tilde{r}_n = \frac{1}{T_p}\int_0^{T_p} \frac{ikh(B_{\mu,0} + B_{\mu,1}\cos(\omega_p t))}{ikh(B_{\mu,0} + B_{\mu,1}\cos(\omega_p t)) - 1} e^{-in\omega_p t} dt, \tag{17}$$

$$\tilde{t}_n = \frac{1}{T_p}\int_0^{T_p} \frac{1}{1 - ikh(B_{\mu,0} + B_{\mu,1}\cos(\omega_p t))} e^{in\omega_p t} dt. \tag{18}$$

Then, one can find explicit relations for the above integrals, by applying contour integration. In the end we find that the coefficients do not depend on $\omega_p$, as expected, and that they are symmetric around the fundamental, i.e. $\tilde{r}_n = \tilde{r}_{-n}$ for $n \neq 0$. The latter is not true in the previous Floquet formalism.

In Fig. 3b, c we depict the harmonics which are reproduced for the plate movement of Fig. 3a, where the bars refer to the $|r_n|$ obtained by means of the Floquet formalism and

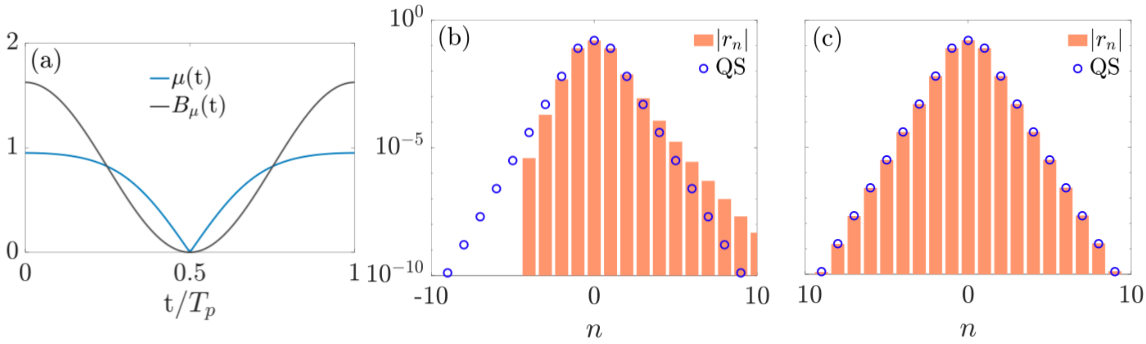


FIGURE 3 – (a) $B_\mu(t)$ and $\mu(t)$ in one characteristic period of oscillation $T_p$. (b,c) Amplitudes $|r_n|$ of the reproduced $n$-harmonics for $\omega\sqrt{h^+/g} = 0.2$, $\omega_p/\omega = 0.25$ (in (b)), and $\omega_p/\omega = 10^{-3}$ (in (c)).

the blue points depict the quasistatic result. In this numerical application, the imposed plate movement was selected to have a high amplitude, with the plate almost reaching the water surface, in order to induce the most reflection in this shallow water regime. The dimensionless frequency of the incident wave is set as $\omega\sqrt{h^+/g} = 0.2$ and $\omega_p = \omega/4$ on panel (b) and $\omega_p = \omega/1000$ on panel (c). It is evident that the closer we move to the limit where $\omega_p/\omega \ll 1$, the better the agreement between the two methods ($|r_n| \to |\tilde{r}_n|$). In addition, the sidebands at $n = \pm 1$ reach roughly 50% the amplitude of the fundamental frequency and the latter is very well captured by the quasistatic limit in both cases of plate oscillation. In order to demonstrate the robustness of the quasistatic approximation in this study, we focus on the fundamental harmonic $n = 0$. We consider two limit values of $\mu(t)$, specifically the maximum height $\mu_{max} = 0.95$ and the mean value of $\mu(t)$, $\mu_{mean} = 0.698$, and wish to compare $|r_0|$ for these static cases versus for the time-varying barrier. As we can see in Fig. 4, $|R_{sw,\mu_{mean}}| < |r_0| < |R_{sw,\mu_{max}}|$ for all values of $\omega$, while $|r_0|$ is practically unaffected by the changes in $\omega_p$. This means that as long as we remain in the shallow water approximation, one can efficiently describe the system with respect to the reflection coefficient of the fundamental harmonic with the explicit relation of the quasistatic approximation.

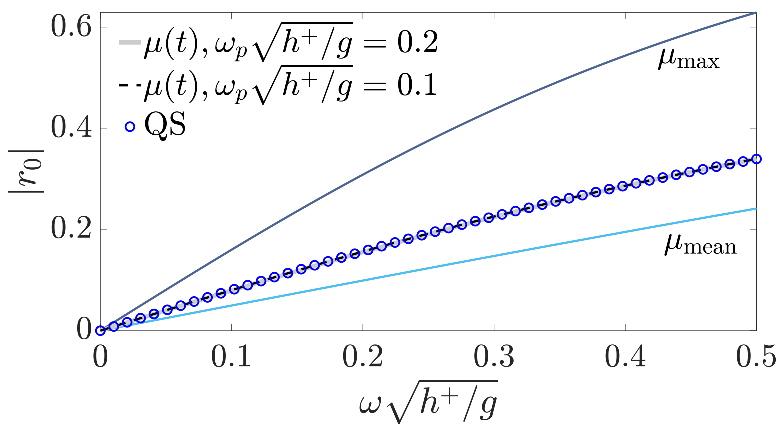


FIGURE 4 – Reflection coefficient of a non-oscillating plate for two different heights $\mu_{mean} = 0.698$, $\mu_{max} = 0.95$ and the reflection coefficient of the fundamental harmonic for the plate oscillation of Fig. 3a when $\omega_p\sqrt{h^+/g} = 0.2, 0.1$ and the result of the quasistatic approximation (Eq. (17)).

### II – 2.2 Experimental observation in intermediate depth

In the experimental part of this work we impose a harmonic plate oscillation $\mu(t)$ and explore the generation of harmonics in the intermediate depth regime. For this purpose, a channel of 3m length and 8cm width is used, which has also a narrow slit in the middle enabling the plate to vertically cross over the channel bottom while being externally connected to a linear motor (see Fig. 5a). The movement of the motor can be very accurately controlled with a software and the plate motion is isolated in a outside cavity attached to the channel. The ongoing experiments have indeed shown that harmonics are generated at frequencies $f_0 \pm n f_p$, $n \in Z$, with $f_0$ the frequency of the incident wave and $f_p$ the frequency of plate oscillation. More precisely, in Fig. 5b we illustrate the Fourier transform of the transmitted field measured with a point laser at a given location ($x = +30$cm) with $f_0 = 2.5$Hz, $f_p = 0.5$Hz, $h^+ = 4$cm and $h_p \in [1.1, 3.6]$cm. Notice that the incident wave is amplified by 35% for an oscillating barrier.

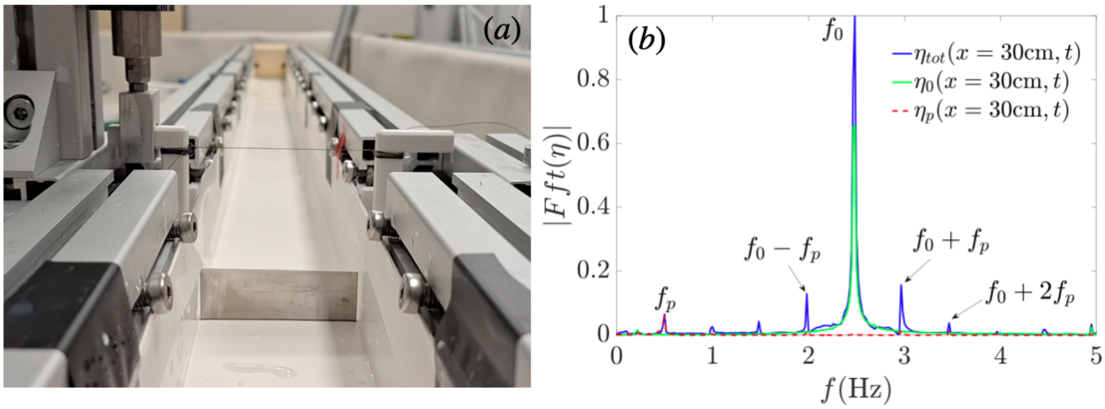


Figure 5 – (a) The vertical plate inside the water channel. (b) Fft of the transmitted field measured at $x = +30$cm when the plate is static at position $h_p = 3.6$cm (green line), when the plate is oscillating (blue line) and when the plate oscillates with no incident wave (red dotted line).

## III – Time-varying plate array

### III – 1 Theory and Numerics

Let us now consider a periodic array of infinitely thin plates sitting on the fluid bottom in a two-dimensional (2D) unbounded domain, as illustrated in Fig. 6a. It has been already shown [10] that the plate array acts as an anisotropic medium for water waves in the long wavelength approximation and that in this limit the wave dynamics can be effectively described by an anisotropic 2D wave equation :

$$\frac{\partial^2 \eta}{\partial t^2} - g\nabla \cdot (\mathrm{h}\nabla\eta) = 0, \quad \mathrm{h} = \begin{pmatrix} h_x & 0 \\ 0 & h_y \end{pmatrix}, \tag{19}$$

$h_x < \langle h^{-1} \rangle^{-1}, h_y = \langle h \rangle^{-1}, \langle h \rangle = \varphi h^- + (1-\varphi)h^+$, with $h^+$, $h^-$, $\varphi$ defined in Fig. 6a and $\eta$ signifying the surface elevation. Then, the time-variation is introduced in the following way : We consider a flat and horizontal fluid bottom for $t < 0$, which is switched into a structured bottom profile at $t = 0$ (and vice-versa). This means that the medium is modified from isotropic to anisotropic at $t = 0$, so that $h(t < 0) = h^+, h(t > 0) = (h_x, h_y)$.

For a medium switch in time, occuring much faster than the wave period, the wave sees a time interface from which it is scattered. When additionally the new medium is anisotropic, the wave is deflected since the angle of energy flow $\theta_S$ is different than the angle of the wave vector $\theta$. As it is stated in [4], one can derive the reflection $R$ and transmission $T$ coefficients as well as $\theta_S$ for the wave scattering by a time interface. These expressions are found to be

$$R = \frac{\omega_1 - \omega_0}{2\omega_1}, \quad T = \frac{\omega_1 + \omega_0}{2\omega_1}, \quad \theta_S = \tan^{-1}\left(\tan(\theta)\frac{h_y}{h_x}\right). \tag{20}$$

with $\omega_0 = k\sqrt{gh^+}, \omega_1 = k\sqrt{g\left(h_x \cos^2\theta + h_y \sin^2\theta\right)}$. By solving Eq. (19) numerically, for $\mathrm{h} = \mathrm{h}(t)$, we obtain the scattering coefficients as a function of the incident angle, which show excellent agreement with the theoretical expressions of Eq. (20) (see Fig. 6b ). In Fig. 6c we illustrate a case of wave packet deviation, where $l = 0.8\text{cm}, \varphi = 0.0625, h^+ = 2\text{cm}, h^- = 0.5\text{cm}, f_0 = 6\text{Hz}, \theta = 20^o$.

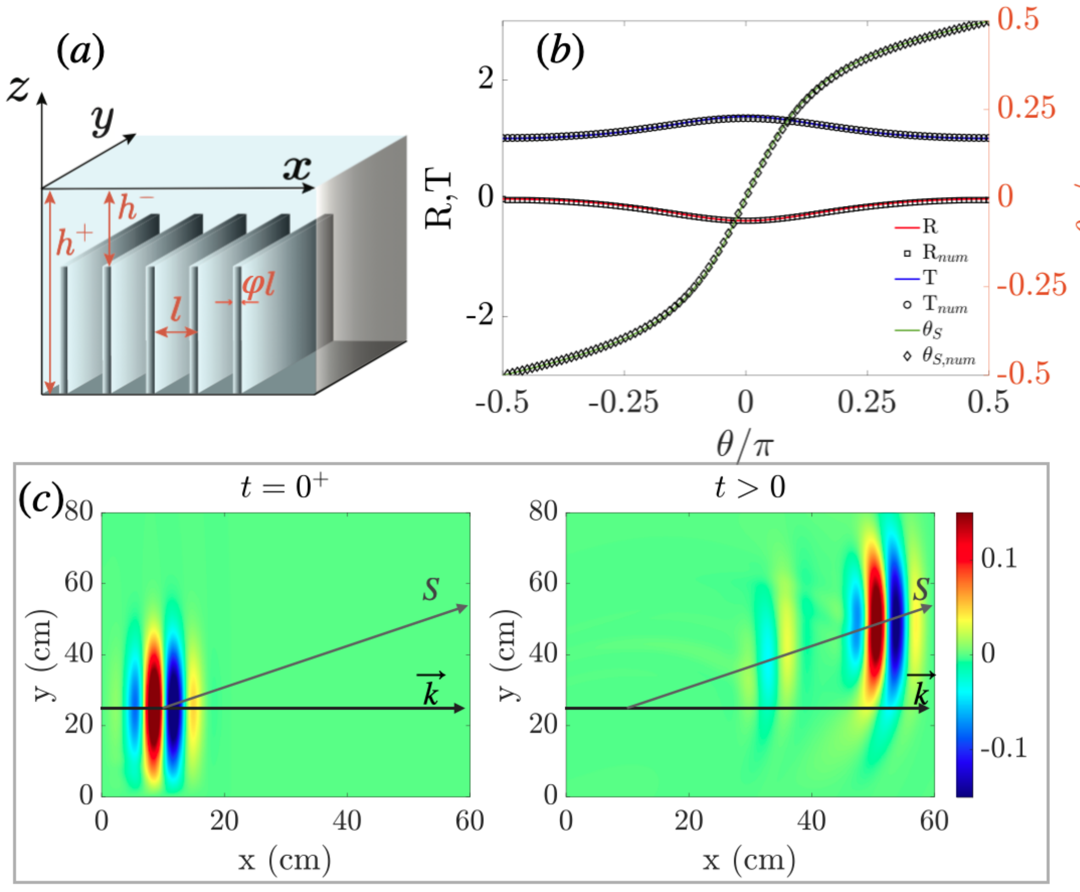


FIGURE 6 – (a) Schematic representation of the submerged plate array. (b) Scattering coefficients $R$, $T$ and the angle of energy flow $\theta_S$ as a function of angle of incidence $\theta$ given from Eq. (20) (in continuous lines) along with the ones computed by solving Eq. (19) with $\mathrm{h} = \mathrm{h}(t)$ switching to an anisotropic medium at $t = 0$ with characteristics $l = 0.8\text{cm}, \varphi = 0.0625, h^+ = 2\text{cm}, h^- = 0.5\text{cm}$ (black symbols). (c) Numerical demonstration of the wavepacket deflection for the previous parameters of the plate array and with $\theta = 20^o, f_0 = 6\text{Hz}$.

### III – 2 Experiment

In the experimental part of this project, we have designed a setup which allows the plate array to be mounted at the fluid bottom (or to descend) at a given time by means of a mechanical switch. More precisely, a circular disc of 55cm diameter is pierced into slits from where the plate array can pass vertically in order to switch the topography from a constant one to a structured one (and vise versa). Below this disc there is a whole mechanism mounted inside the water tank which guides this upward movement of the plate array. Fig. 7 displays the top view of the experimental setup, specifically the topography change when we pass from the initial anisotropic medium to the isotropic one at $t = 0$.

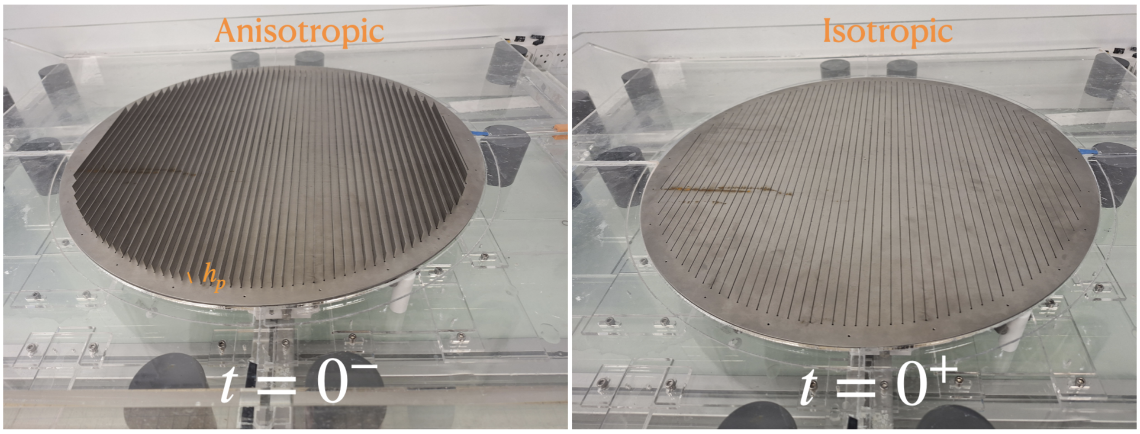


FIGURE 7 – Experimental setup showing the bottom profile before and after the switch at $t = 0$.

For this configuration, i.e. where the plate array is abruptly dropped at $t = 0$, experimental results have been obtained which clearly show the deflection of the wavepacket (see Fig. 8). For negative times, the generated wavepacket travelling in the anisotropic medium deviates downwards according to the theoretical prediction. Then, at $t = 0$ when the medium is switched into an isotropic one, the wavepacket changes trajectory and moves straight, as shown with the gray arrow.

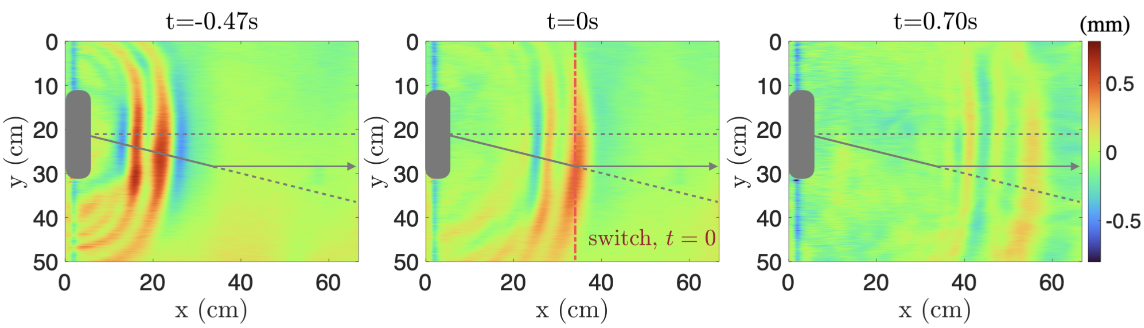


FIGURE 8 – Experimental demonstration of the wavepacket deviation at three different snapshots ($t = -0.47$s, $t = 0$s and $t = 0.7$s) when the medium is switched from anisotropic to isotropic at $t = 0$. The wavepacket follows the path illustrated with the gray arrow.

## IV – Conclusion and perspectives

We have discussed two problems which aim to control water waves using time-varying vertical plates as metamaterials. In the first one, we tackled the scattering problem of a monochromatic wave from an infinitely thin plate, which is vertically oscillating. We proposed a Floquet theory approach in the shallow water approximation and showed that in that limit $n$-harmonics are generated around the fundamental frequency of the incident wave $\omega$, with the first sidebands ($n = \pm 1$) representing an important percentage of the reflected amplitude of the fundamental, regardless of the frequency of oscillation $\omega_p$. In addition, for a slowly oscillating plate, i.e. when $\omega_p \ll \omega$, we implemented a quasistatic adiabatic approximation and found explicit relations for the scattering coefficients, which match the ones recovered by the Floquet theory approach as $\omega_p \to 0$. This quasistatic approximation appears to be particularly robust, as it captures perfectly the reflection coefficient of the fundamental frequency for any $\omega_p$. Furthermore, the generation of harmonics has been also observed experimentally in the intermediate depth regime for a harmonically oscillating plate and the experimental works are continuing with the aim to fully quantify and characterize the wave dynamics in any water depth regime.

In the second part, we demonstrated how one can deflect a wavepacket travelling in a two-dimensional space by combining an anisotropic medium with time-variation, which is the water-wave analog of the temporal aiming first introduced in optics [4]. In this context, we used an array of infinitely thin plates which played the role of an effective anisotropic medium in the shallow water approximation, and tuned in the time variation by lifting or dropping the plates at the fluid bottom at a given time so as to switch the effective medium. We retrieved numerically the scattering coefficients and the angle of the energy flow for any angle of incidence by solving the anisotropic two-dimensional wave equation, which agree with the analytical predictions. Finally, in the experimental part of this project we showed an example of wavepacket deviation, starting from the anisotropic medium and switching it to isotropic at $t = 0$. This first observation proves that the wavepacket can ideed be guided in space by exploiting this plate array metamaterial along with time-variation, and more exploration is currently being done with our experimental setup in order to find new aspects for the control the wavepacket propagation.

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