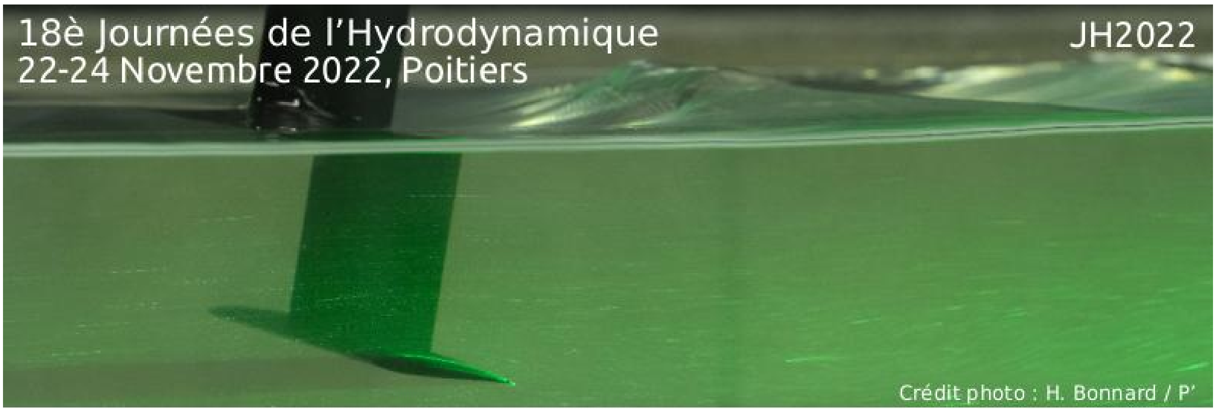

Un nouveau modèle d'eau peu profonde d'ordre élevé avec une structure Hamiltonienne canonique

A new high-order shallow-water model with canonical Hamiltonian structure

CH. PAPOUTSELLIS[(1)]
christos.papoutsellis@ensta-bretagne.fr

[(1)] Institut de Recherche Dupuy de Lôme, École Nationale Supérieure de Techniques Avancées Bretagne, Centre National de la Recherche Scientifique, UMR6027

Résumé

Dans ce travail, nous dérivons un nouveau modèle de vagues en utilisant un ansatz pour le potentiel de vitesse inspiré de la théorie des eaux peu profondes. Le potentiel est représenté par une série avec des coefficients fonctionnels inconnus qui dépendent de la variable horizontale et du temps et des fonctions verticales (polynômes) qui correspondent à celles qui apparaissent dans une expansion asymptotique. Nous montrons que les équations dérivées ont une structure Hamiltonienne canonique non-locale en accord avec la formulation Hamiltonienne du problème complet. Nous discutons de la relation avec les modèles existants et fournissons quelques résultats numériques.

Summary

In this work, we derive a new water-wave model by using an ansatz for the velocity potential inspired by shallow-water theory. The velocity potential is represented by a series with unknown functional coefficients that dependent on the horizontal variable and time and vertical functions (polynomials) that match the ones appearing in an asymptotic expansion. We show that the derived equations have a canonical non-local Hamiltonian structure in accordance with the Hamiltonian formulation of the full wate-wave problem. We discuss the relation with existing models and provide some numerical results.

I – Introduction

The majority of water wave models currently in use in oceanographic and coastal engineering applications are based on asymptotic expansions or expansions around a plane vertical level (e.g. the still water level). This approach leads to "simple" evolution equations defined only on the horizontal plane. Thus, the free-boundary character of the full problem and its internal kinematics are simplified at the cost of limited range of applicability. Since the early works of Boussinesq in the 1900s several models have been proposed which are reviewed in [7, 18], for example. However, the need of improvement still exists [26, 19].

A popular and extensively studied model based on an asymptotic expansion in terms of the shallowness is provided by the Serre-Green-Naghdi (SGN) equations [21, 27] which are two evolution partial differential equations (PDEs). The SGN equations have been extensively studied and have led to several numerical schemes that can also treat breaking and run-up [6, 14]. One of the difficulties in the numerical solution of the SGN equations is that they involve the inversion of a linear operator that relates the evolving variables [24, 13, 22].

A different class of models is obtained by using eigenfunction expansions [5, 34, 4] or Chebyshev expansions [40] to exactly prescribe the vertical structure of the potential and derive equations for the introduced unknown horizontal coefficients. These models have the structure of the Zakharov-Craig-Sulem formulation [41, 10] where the Laplace equation is replaced by a system of spatial PDEs on the horizontal plane whose dimension depends on the order of truncation of the expansions. No smallness assumptions is needed a priory for their derivation and their precision increases with the order of truncation. They can treat very demanding cases, including breaking waves, if a sufficient number of terms (or PDEs) is kept in their truncated version (typically 4-7) [33, 2, 31, 3, 36, 35, 37, 42].

Another approach is to use an appropriately constructed ansatz for the vertical structure of the potential. Isobe and Kakinuma (IK) [16, 17] used the vertical polynomials in the Boussinesq-Rayleigh asymptotic expansion. This class of models has recently attracted attention possibly due to their non-trivial mathematical structure [29, 15, 30, 9, 11]. A method for their numerical solution is proposed in [1]. Klopman et al. [20] proposed different vertical polynomials or hyperbolic cosine functions that lead directly to Hamiltonian model equations with their most recent version, involving a dispersion improvement, showing very good properties [23].

In this work, we consider an ansatz for the velocity potential where the vertical structure is identified with the polynomials obtained by the asymptotic expansion of [21]. This expansion is obtained directly in terms of free-surface quantities from the Dirichlet-to-Neumann problem of the ZCS formulation. By invoking Luke's variational pricniple for the derivation of the model equations we directly obtain the equations in Hamiltonian form where in the simplest non-trivial flat-bottom case the Laplace equation is approximated by a single elliptic PDE on the horizontal plane. The derivation does not require a smallness assumption on the free surface and the resulting equations have at most second-order spatial derivatives no matter the number of terms that are kept in the ansatz.

II – The water-wave problem

We consider the motion of water waves in a cartesian coordinate system OXz with $X = (x_1, x_2)$ and the vertical z-axis pointing upwards with $z = 0$ corresponding to still water level. The fluid motion under the assumption of potential flow is described in terms of the velocity potential $\Phi = \Phi(X, z, t)$ and the free surface elevation $\eta = \eta(X, t)$ by the following set of equations [38]

$$\boldsymbol{\Delta}\Phi = \Delta\Phi + \partial_z^2\Phi = 0, \quad -h \leq z \leq \eta, \tag{1a}$$

$$N_h\boldsymbol{\cdot}\boldsymbol{\nabla}\Phi = -\nabla h \cdot \nabla\Phi - \partial_z\Phi = 0, \quad z = -h, \tag{1b}$$

$$\partial_t\eta - N_\eta\boldsymbol{\cdot}\boldsymbol{\nabla}\Phi = \partial_t\eta + \nabla\eta \cdot \nabla\Phi - \partial_z\Phi = 0, \quad z = \eta, \tag{1c}$$

$$\partial_t\Phi + \frac{1}{2}(\boldsymbol{\nabla}\Phi)^2 + g\eta = \partial_t\Phi + \frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}(\partial_z\Phi)^2 + g\eta = 0, \quad z = \eta, \tag{1d}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, $\boldsymbol{\nabla} = (\nabla, \partial_z)$, $\nabla = (\partial_{x_1}, \partial_{x_2})$, $N_\eta = (-\nabla\eta, 1)$ and $N_h = (-\nabla h, -1)$ and g is the acceleration of gravity. We also assume that $h > 0$, $\eta + h > 0$ and that η and $\nabla\Phi$ vanish as $|X| \to \infty$.

Luke [25] showed that the equations in (1) are obtained as the *Euler-Lagrange* (EL) equations of the functional given by the space-time integral of the fluid pressure,

$$\mathcal{S}(\eta, \Phi) = \int\int\int_{-h}^{\eta} \left[\partial_t\Phi + \frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}(\partial_z\Phi)^2 + gz\right] \mathrm{d}z\, \mathrm{d}X\, \mathrm{d}t\,. \tag{2}$$

The usefulness of Luke's variational principle, $\delta\mathcal{S} = 0$, stems from the fact that choosing an ansatz for Φ in terms of new unknowns the variations will provide their governing equations [5, 8, 33, 34].

Zakharov [41] introduced the free-surface velocity potential $\psi = \Phi(X, z, \eta(X, t))$ and showed that free-surface kinematic and dynamic conditions (1c) and (1d) can be re-written in the Hamiltonian form,

$$\begin{aligned} \partial_t\eta &= \delta_\psi H, \\ \partial_t\psi &= -\delta_\eta H, \end{aligned} \qquad H = \frac{1}{2}\int\int_{-h}^{\eta} (\boldsymbol{\nabla}\Phi)^2\, \mathrm{d}z\, \mathrm{d}X + \frac{1}{2}g\int_X \eta^2 \mathrm{d}X, \tag{3}$$

where the Hamiltonian H equals to the total energy of the fluid with Φ satisfying the internal kinematics equations (1a), (1b), and δ_η and δ_ψ denote variational derivatives with respect to η and ψ respectively. The canonical Hamiltonian structure (3) is non-local in the sense that the Hamiltonian and the right hand sides of the evolution PDEs contain the non-local Dirichlet-to-Neumann operator determined by the solution of the Laplace equation in the entire domain, as opposed to a usual differential operator which is defined locally at any point. Below we present a model derived from (2) that preserves the Hamiltonian structure (3).

III – Variational derivation of a model equation

We exploit Luke's variational principle in order to derive model equations based on vertical series representations of the velocity potential in the form

$$\Phi^{\mathrm{a}}(X, z, t) = \sum_n \varphi_n(X, t) Z_n(z; \eta(X, t), h(X)) \tag{4}$$

where $Z_n = Z_n(z; \eta(X, t), h(X))$ have a prescribed dependence on z, h and η and $\varphi_n = \varphi_n(X, t)$ are new unknown functions for which equations are to be derived. Motivated by shallow-water theory [21], we choose the following representation of the velocity potential in the case of flat bottom ($\partial_{x_i} h = \Delta h = 0$, $N_h = (0, -1)$)

$$\Phi^{\mathrm{a}} = \psi + \varphi Z(z; \eta, h), \quad \text{with} \quad Z(z; \eta, h) = \frac{1}{2}(z - \eta)^2 + (\eta + h)(z - \eta)\,. \tag{5}$$

and we extremize Luke's functional for functions of the form (5). Substituting (5) into Luke's functional (2) and taking the variations with respect to the φ, ψ and η (see e.g. [32, 33, 34]) we obtain three Euler-Lagrange equations. After some manipulations, these equations can be written in the following form

$$\partial_t \eta + \nabla \cdot \left[H \nabla \psi - \frac{1}{3} \nabla (H^3 \varphi) \right] = 0, \tag{6a}$$

$$\partial_t \psi + g\eta + \frac{1}{2} (\nabla \psi)^2 - \left(\frac{3}{2} |\nabla \eta|^2 - \frac{1}{2} \right) (H\varphi)^2 + \left[H \Delta \psi - \frac{1}{3} \Delta (H^3 \varphi) \right] H\varphi = 0, \tag{6b}$$

$$\frac{2}{5} \Delta \varphi + \frac{2}{H} \nabla \eta \cdot \nabla \varphi + \left(\frac{\Delta H}{H} + \frac{(\nabla \eta)^2}{H^2} - \frac{1}{H^2} \right) \varphi = \frac{\Delta \psi}{H^2}. \tag{6c}$$

Eqs. (6a), (6b) are two evolution PDEs on (η, ψ) and contain φ which is determined, at every time t, in terms of (η, ψ) by solving the spatial elliptic PDE (6c).

IV – Hamiltonian structure

In order to establish the Hamiltonian structure of (6) we need to introduce a Hamiltonian defined on (η, ψ). This is done be assuming that φ is expressed in terms of (η, ψ) by solving (6c). We then may introduce the mapping $\psi \rightarrow \phi[\eta]\psi := \varphi$ and write the Hamiltonian as

$$\mathcal{H}^{\mathrm{a}}(\eta, \psi) = \frac{1}{2} \int \int_{-h}^{\eta} \big(\boldsymbol{\nabla} (\psi + (\phi[\eta]\psi) Z) \big)^2 \, \mathrm{d}z \, \mathrm{d}X + \frac{1}{2} g \int \eta^2 \, \mathrm{d}X \tag{7}$$

Next, we must calculate the variational derivatives of $\mathcal{H}^{\mathrm{a}}$ which in turn requires the Fréchet derivatives of $\phi[\eta]\psi$ with respect to η and ψ for which explixit expressions are not available, in general; they are defined in terms of the BVP (6c) and its derivative with respect to η and are denoted by $D_\psi \mathcal{H}(\delta\psi)$ and $D_\eta \mathcal{H}(\delta\eta)$ [11]. Using this fact, we can show that

$$D_\psi \mathcal{H}^{\mathrm{a}}(\eta, \psi)(\delta\psi) = \int \partial_\psi \mathcal{H}^{\mathrm{a}} \, \delta\psi \, \mathrm{d}X \,, \quad D_\eta \mathcal{H}^{\mathrm{a}}(\eta, \psi)(\delta\eta) = \int \partial_\eta \mathcal{H}^{\mathrm{a}} \, \delta\eta \, \mathrm{d}X \,. \tag{8}$$

where $\partial_\psi \mathcal{H}^{\mathrm{a}}$ and $\partial_\eta \mathcal{H}^{\mathrm{a}}$ denote the variational derivatives that must be calculated. As an example, we sketch here the derivation of the first equation in (8). For all $\delta\psi$, we have

$$D_\psi \mathcal{H}^{\mathrm{a}}(\eta, \psi)(\delta\psi) = \int \int_{-h}^{\eta} \big(\boldsymbol{\nabla} (\psi + (\phi[\eta]\psi) Z) \big) \big(\delta\psi + (\phi[\eta]\delta\psi) Z \big) \, \mathrm{d}X$$

Using Green's identity, the fact that $[Z]_\eta = 0$ and (6c) we find the expression for $\delta_\psi \mathcal{H}^a$. The calculation of $D_\eta \mathcal{H}(\delta\eta)$ can be performed along the same lines. Then, one can verify that the system (6a), (6b) is equivalent with the Hamilton's equations

$$\partial_t \eta = \delta_\psi \mathcal{H}^a, \tag{9}$$

$$\partial_t \psi = -\delta_\eta \mathcal{H}^a. \tag{10}$$

V – Linear properties and relation with other models

In order to gain some insight on the physical properties of (6), we linearise around $(\eta, \psi) = (0, 0)$ and look for plane wave solutions in the form $(\eta, \psi, \varphi) = (\tilde{\eta}, \tilde{\psi}, \tilde{\varphi}) e^{i(\kappa x - \omega t)}$. Eq. (6c) can be solved for φ and we obtain from (6a) and (6b) the dispersion relation

$$\omega^2 = gh\kappa^2 \frac{1 + \frac{1}{15}h^2\kappa^2}{1 + \frac{2}{5}h^2\kappa^2}. \tag{11}$$

Eq. (11) coincides with the dispersion relation of the parabolic model in [20] and of the improved Green-Naghdi equations in [6, Section 2.6] with $a = 6/5$. It also coincides with the dispersion relation of the Isobe-Kakinuma (IK) [29]. The IK model is obtained from Luke's variational principle by choosing

$$\Phi^{\mathrm{IK}}(X, z, t) = \phi_0 Z_0 + \phi_1 Z_1 + \cdots + \phi_K Z_K, \quad Z_n = (z - h)^{2n}, \tag{12}$$

as an ansatz for Φ in the flat bottom case. For $K = 1$, the IK model is a system of evolution PDEs in (η, ϕ_0, ψ_1) with an unusual structure which is not convenient for numerical calculations. However, it can be transformed into a canonical Hamiltonian system by introducing the free-surface velocity potential $\psi = \phi_0 + H^2\phi_1$ [11]. By using this relation and straightforward manipulations, it can be shown that this transformed Hamiltonian system is actually equivalent to (6). It should be noted that the IK model is a $O(\sigma^6)$ approximation of the water-wave system and this is expected to be true for (6). Since these type of models are derived using ansatzes containing the vertical structure of asymptotic expansions of the potential, it is natural to ask if they can be reduced to the usual shallow-water models. Note that both models lead to the canonical form of the non-linear shallow water equations if only the first term is kept in the invoked ansatzes. We briefly sketch here a quick way to to simplify (6) by reducing its precision in σ. We note that ψ scales as $L\sqrt{gh_0}$, φ as $L\sqrt{gh_0}/h_0^2$ by construction, and the first horizontal derivatives are first order in σ. We can then obtain from (6) that $\varphi = -\Delta\psi + O(\sigma^2)$ thus recovering the $O(\sigma^2)$-asymptotic expansion of [21]. Using this result we can show that (6a) and (6b) lead to the canonical equations written in terms of (η, ψ) appearing in [39] and [28]. These equations lead to the usual SGN system in terms of η and the vertically-integrated horizontal velocity.

VI – A numerical example

As a numerical application of (6), we numerically solve the initial value problem in a periodic domain by using as initial conditions highly accurate solitary wave solutions of the irrotational Euler equations obtained in [12]. At every time step we solve (6c) with a spectral Fourier collocation method and (6a),(6b) with a Fourier pseudo-spectral method in conjunction with the classical fourth-order Runge-Kutta method. We also compare our results with the Fourier pseudo-spectral solution of the SGN equations [13]. The horizontal domain is $[-80, 80]$ and the depth is $h = 1$. The number of Fourier coefficients and physical points is $N = 1024$ and the solitary wave is initially centered at $x/h_0 = -50$. We consider three increasing relative heights $a/h_0 = 0.15$, 0.30 and 0.45 and let the waves propagate for 80 times the depth with a time step $dt = 0.5dx/c$ where c is the phase speed of the exact solitary wave. We plot the results in figure 1. We see that (6) performs very well stably propagating the exact solitary wave of the Euler equations. Slight differences at the phase speed and the maximum height are visible at the scale of the figure for $a/h_0 = 0.45$. The SGN also maintain the solitary wave shape though differences are more pronounced as a/h_0 increases. It should be noted that the advancement in time of both the SGN system and (6), requires the solution of a linear equation a every time-step but detailed efficiency comparisons are out of the scope of this work.

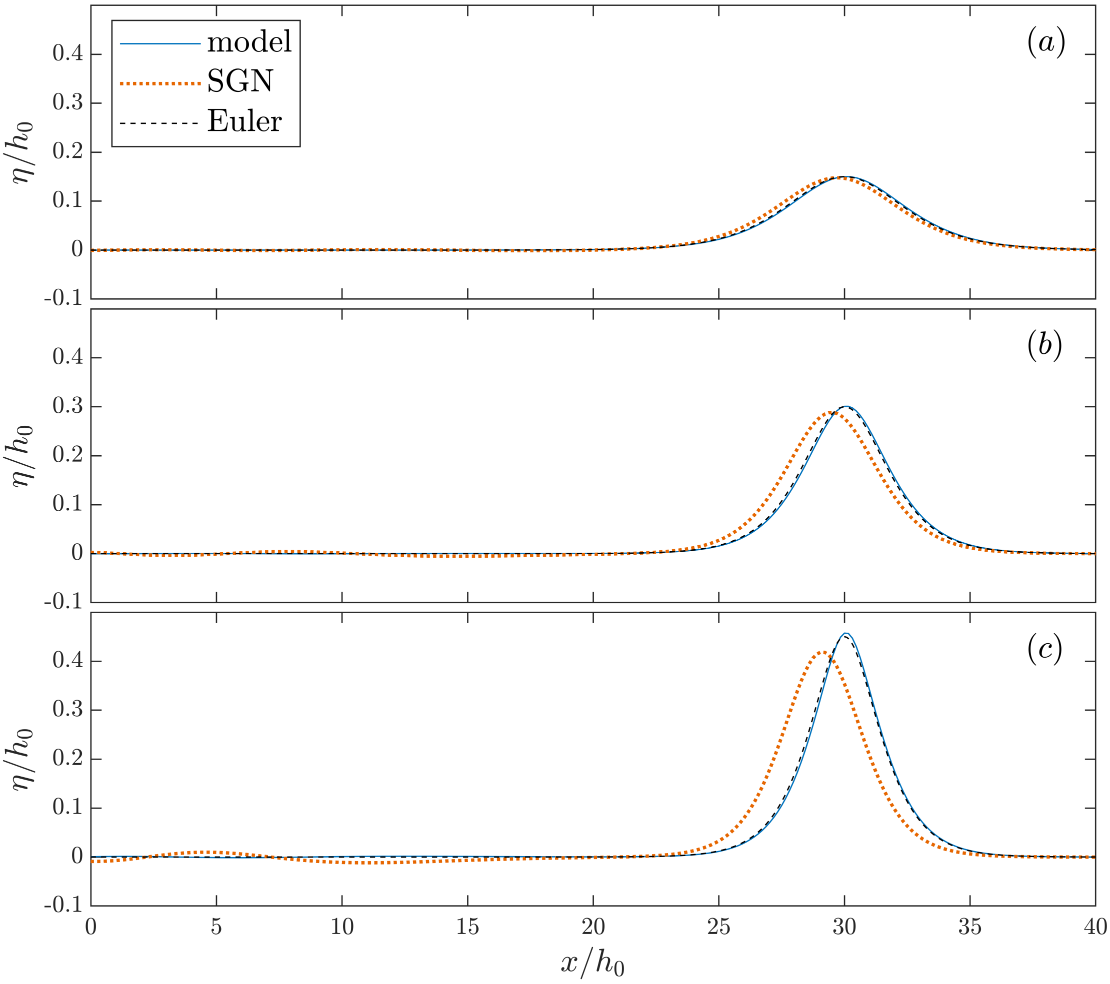

Figure 1: Propagation of a solitary wave of relative height 0.15 (a), 0.3 (b) and 0.45 (c). Blue line corresponds to (6).

VII – Conclusions and perspectives

We presented a wave model derived from Luke's variational principle by using an ansatz for the velocity potential inspired from the shallow-water asymptotic expansion of [21]. No simplification is made in terms of the non-linearity. The equations exhibit a natural non-local canonical Hamiltonian structure inherited from the ZCS formulation of water waves. The proposed model is equivalent to the IK model which is a $O(\sigma^6)$-approximation of the ZCS formulation and can be further reduced to the SGN equations. First numerical results are very promising and suggest that this model could be useful for applications in which strong dispersive and non-linear effects are present. The approach proposed here can be extended in a straightforward way to higher-orders and variable bottom.

Acknowledgements

A part of this work was carried out during the PhD thesis of the author with Professor Gerassimos Athanassoulis which was funded by the National Technical University of Athens (2012-2016). This work has also benefited from collaboration with Professor Michel Benoit.

References

[1] S. Abohadima and M. Isobe. Linear and nonlinear wave diffraction using the nonlinear time dependent mild slope equations. *Coastal Engineering*, 37(2):175–192, 1999.

[2] G. Athanassoulis, K. Belibassakis, and Ch. Papoutsellis. An exact Hamiltonian coupled-mode system with application to extreme design waves over variable bathymetry. *J. Ocean Eng. Mar. Energy*, 3:373–383, 2017.

[3] G. Athanassoulis, C. Mavroeidis, P. Koutsogiannakis, and Ch. Papoutsellis. A numerical study of the run-up and the force exerted on a vertical wall by a solitary wave propagating over two tandem trenches. *J. Ocean Eng. Mar. Energy*, 5:311–331, 2019.

[4] G. A. Athanassoulis and Ch. Papoutsellis. Exact semi-separation of variables in waveguides with non-planar boundaries. *Proc. R. Soc. A*, 473(2201):20170017, 2017.

[5] K. Belibassakis and G. Athanassoulis. A coupled-mode system with application to nonlinear water waves propagating in finite water depth and in variable bathymetry regions. *Coast. Eng.*, 58:337–350, 2011.

[6] P. Bonneton, F. Chazel, D. Lannes, F. Marche, and M. Tissier. A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model. *J. Comp. Phys*, 230:1479–1498, 2011.

[7] M. Brocchini. A reasoned overview on Boussinesq-type models: the interplay between physics, mathematics and numerics. *Proc. R. Soc. A*, 469(20130496), 2013.

[8] D. Clamond and D. Dutykh. Practical use of variational principles for modelling water waves. *Physica D*, 241:25–36, 2012.

[9] M. Colin and T. Iguchi. Solitary wave solutions to the isobe-kakinuma model for water waves. *Studies in Applied Mathematics*, 145(1):52–80, 2020.

[10] W. Craig and C. Sulem. Numerical simulation of gravity waves. *J. Comp. Phys.*, 108:73–83, 1993.

[11] V. Duchêne and T. A. Iguchi. A Hamiltonian Structure of the Isobe–Kakinuma Model for Water Waves. *Water Waves*, 3:193–211, 2021.

[12] D. Dutykh and D. Clamond. Efficient computation of steady solitary gravity waves. *Wave Motion*, 51(1):86–99, 2014.

[13] D. Dutykh, D. Clamond, P. Milewski, and D. Mitsotakis. Finite volume and pseudo-spectral schemes for the fully nonlinear 1D Serre equations. *Eur. J. Appl. Math*, 24(5):761–787, 2013.

[14] A. G. Filippini, M. Kazolea, and M. Ricchiuto. A flexible genuinely nonlinear approach for nonlinear wave propagation, breaking and run-up. *J. Comput. Phys.*, 310:381–417, 2016.

[15] T. Iguchi. A Mathematical Justification of the Isobe–Kakinuma Model for Water Waves with and without Bottom Topography. *Math. Fluid Mech.*, 20:1985–2018, 2018.

[16] M. Isobe and S. Abohadima. Time-dependent mild-slope equations for random waves. *Coast. Eng.*, pages 285–299, 1994.

[17] T. Kakinuma. *A set of fully nonlinear equations for surface and internal gravity waves*. Environmental Studies. WIT Press, 2001.

[18] J. T. Kirby. Boussinesq Models and Their Application to Coastal Processes across a Wide Range of Scales. *J. Waterw. Port Coast. Ocean Eng*, 142(6):03116005, 2016.

[19] J. T. Kirby. A new instability for Boussinesq-type equations. *J. Fluid Mech.*, 894:F1, 2020.

[20] G. Klopman, B. V. Groesen, and M. Dingemans. A variational approach to boussinesq modelling of fully nonlinear water waves. *J. Fluid Mech*, 657:36–63, 2010.

[21] D. Lannes and P. Bonneton. Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation. *Phys. Fluids*, 21(016601), 2009.

[22] D. Lannes and F. Marche. A new class of fully nonlinear and weakly dispersive Green–Naghdi models for efficient 2D simulations. *J. Comput. Phys.*, 282:238–268, 2015.

[23] C. Lawrence, D. Adytia, and E. van Groesen. Variational Boussinesq model for strongly nonlinear dispersive waves. *Wave Motion*, 76:78–102, 2018.

[24] O. Le Métayer, S. Gavrilyuk, and S. Hank. A numerical scheme for the Green–Naghdi model. *J. Comput. Phys.*, 229(6):2034–2045, 2010.

[25] J. Luke. A variational principle for a fluid with a free surface. *J. Fluid Mech.*, 27:395–397, 1967.

[26] P. A. Madsen and D. R. Fuhrman. Trough instabilities in Boussinesq formulations for water waves. *J. Fluid Mech.*, 889:A38, 2020.

[27] Y. Matsuno. Hamiltonian structure for two-dimensional extended Green-Naghdi equations. *Proc. R. Soc. A*, 472(2190), 2016.

[28] J. Miles and R. Salmon. Weakly dispersive nonlinear gravity waves. *J. Fluid Mech.*, 157:519–531, 1985.

[29] Y. Murakami and T. Iguchi. Solvability of the initial value problem to a model system for water waves. *Kodai Math. J.*, 38:470–491, 2015.

[30] R. Nemoto and T. Iguchi. Solvability of the Initial Value Problem to the Isobe–Kakinuma Model for Water Waves. . *J. Math. Fluid Mech.*, 20:632–653, 2018.

[31] T. Papathanasiou, Ch. Papoutsellis, and G. A. Athanassoulis. Semi-explicit solutions to the water-wave dispersion relation and their role in the non-linear Hamiltonian coupled-mode theory. *J. Eng. Math.*, 114(1):87–114, 2018.

[32] Ch. Papoutsellis. *Nonlinear water waves over varying bathymetry: theoretical and numerical study using variational methods*. PhD thesis, National Technical University of Athens, 2016. http://dx.doi.org/10.26240/heal.ntua.2614.

[33] Ch. Papoutsellis and G. Athanassoulis. A new efficient hamiltonian approach to the nonlinear water-wave problem over arbitrary bathymetry. https://arxiv.org/abs/1704.03276, 2017.

[34] Ch. Papoutsellis, A. Charalampopoulos, and G. Athanassoulis. Implementation of a fully nonlinear Hamiltonian Coupled-Mode Theory, and application to solitary wave problems over bathymetry. *Eur. J. Mech. B*, 72:199–224, 2018.

[35] Ch. Papoutsellis, B. Simon, M. Yates, and M. Benoit. Modelling of depth-induced wave breaking in a fully nonlinear free-surface potential flow model. *Coast. Eng*, 157:103579, 2019.

[36] C. Raoult, M. Benoit, and M. L. Yates. Development and validation of a 3D RBF-spectral model for coastal wave simulation. *J. Comput. Phys.*, 378:278–302, 2019.

[37] B. Simon, Ch. Papoutsellis, M. Yates, and M. Benoit. Modelling of depth-induced wave breaking in a fully nonlinear free-surface potential flow model. *J. Ocean Eng. Mar. Energy*, 5:365—-383, 2019.

[38] J. Stoker. *Water Waves*. Interscience, NY, 1957.

[39] G. B. Whitham. Variational methods and applications to water waves. *Proc. R. Soc. A*, 299(1456):6–25, 1967.

[40] M. Yates and M. Benoit. Accuracy and efficiency of two numerical methods of solving the potential flow problem for highly nonlinear and dispersive water waves. *Int. J. Numer. Methods Fluids*, 77:616–640, 2015.

[41] V. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys*, 9:86–94, 1968.

[42] J. Zhang and M. Benoit. Wave–bottom interaction and extreme wave statistics due to shoaling and de-shoaling of irregular long-crested wave trains over steep seabed changes. *J. Fluid Mech.*, 912:A28, 2021.