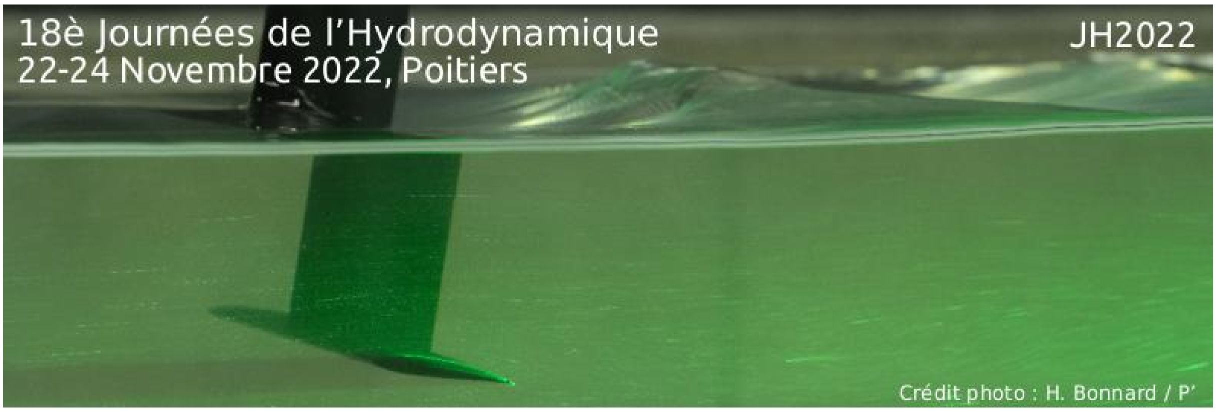

Modèles d'ondes dispersives pour les écoulements pulsatiles dans les vaisseaux viscoélastiques

Nonlinear dispersive wave models for pulsatile flows in viscoelastic vessels

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Résumé

Dans cet exposé, nous allons examiner le problème de l'écoulement pulsatile dans les récipients déformables à parois viscoélastiques. Par souci de simplicité, nous supposons que le flux est axisymétrique et le fluide idéal. À savoir, dans ce travail, nous effectuons la réduction de l'ordre de modèle sous l'hypothèse de longue vague des équations d'Euler axi-symétriques. Nous obtenons ainsi de nouvelles équations de modèles asymptotiques décrivant la propagation d'impulsions à crête longue dans des tubes déformables avec symétrie cylindrique [8].

Les effets supplémentaires dus aux contraintes visqueuses dans les bio-fluides peuvent également être pris en compte. Traditionnellement, c'est un système hyperbolique d'équations qui est utilisé. Dans notre travail, nous proposons diverses extensions faiblement dispersives en présence de la symétrie cylindrique. Nous nous concentrons d'abord sur le régime entièrement non linéaire conduisant à des équations de modèles relativement complexes. Afin de simplifier davantage le système de type Serre obtenu, nous en dériverons sa contrepartie faiblement non linéaire ainsi que les réductions unidirectionnelles du modèle (*e.g.* KdV, équations de type BBM).

les nouveaux systèmes sont étudiés analytiquement en fonction de leurs propriétés caractéristiques de base telles que les symétries, les lois de conservation et les ondes solitaires [17, 4]. Certainement, les propriétés linéaires telles que les relations de dispersion sont discutées par rapport au modèle de base.

Summary

In this talk we are going to consider the pulsatile flow problem in deformable vessels with visco-elastic walls. For the sake of simplicity, we assume the flow to be axi-symmetrical and the fluid ideal. Namely, in this work we perform the model order reduction under the long wave assumption of axi-symmetric Euler equations. In this way we derive new asymptotic model equations describing the propagation of long-crested pulses in deformable pipes with cylindrical symmetry [8].

Additional effects due to viscous stresses in bio-fluids can be also taken into account. Traditionally, it is a hyperbolic system of equations which is being used. In our work we propose various weakly dispersive extensions in the presence of the cylindrical symmetry [9]. We focus first on the fully nonlinear regime leading to relatively complex model equations. In order to simplify further the obtained Serre-type system, we derive its weakly-nonlinear counterpart along with unidirectional model reductions (*e.g.* KdV, BBM-type equations).

The new systems are studied analytically in terms of their basic characteristic properties such as the symmetries, conservation laws and solitary travelling waves [17, 4]. Of course, the linear properties such as the dispersion relations are discussed with respect to the base model.

I – Introduction

A beating heart creates pressure and flow pulsations that propagate as waves through the arterial tree that are reflected at transitions in arterial geometry and elasticity. Indeed, the vascular wall is a living tissue with the presence of muscalar cells which contribute to its mechanical behaviour. The mechanical interaction between fluid and vessel walls makes the flow kind of a complex structure, one of the factors affecting blood flow is the flexible nature of the vessels, the large arteries are deformed by the large blood pressure, and they store energy during the systolic phase to return it in the diastolic phase. We are facing a problem whose complexity is enormous. It is the role of mathematical modeling to find reasonable simplifying assumptions by which major physical characteristics remain present [2, 14, 16, 13, 15]. The mathematical modelling of such flows suggests the use of incompressible and ideal and radially symmetric fluid flow equations known to as the Euler equations.

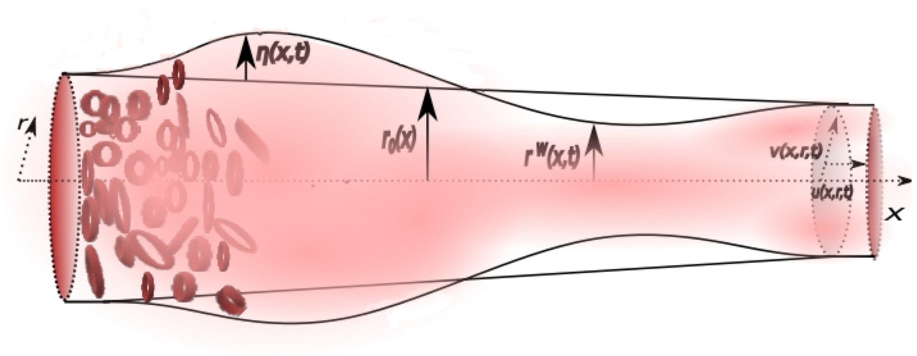

Figure 1 – Sketch of the physical domain of a single vessel segment with elastic and impenetrable wall

These equations written in cylindrical coordinates take the form [10]

$$u_t + u u_x + v u_r + \frac{1}{\rho} p_x = 0, \tag{1}$$

$$v_t + u v_x + v v_r + \frac{1}{\rho} p_r = 0, \tag{2}$$

$$u_x + v_r + \frac{1}{r} v = 0, \tag{3}$$

where $u = u(x, r, t), v = v(x, r, t)$ are the horizontal and the radial velocity of fluid respectively, $p = p(x, r, t)$ is the pressure of the fluid, while ρ is the constant density of the fluid. We denoted $r^w(x, t)$ as the distance of vessel's wall from the center of the

vessel in a cross section, and it depends on x and t while the radius of the vessel at rest is the function $r_0(x)$. A sketch of the physical domain of this problem is given by figure 1. In general, deformation of the wall will be a function of x and t. If we denote the radial displacement of the wall by $\eta(x, t)$ then the vessel wall radius can be written as $r^w(x, t) = r_0(x) + \eta(x, t)$.
The prevailing equations (1 – 3) along with initial and boundary conditions form a closed system. A compatibility condition is also applied at the center of the vessel. Specifically, we assume that

$$v(x, r, t) = 0, \quad \text{for} \quad r = 0,$$

The form of the impermeability condition on the vessel wall can be written as

$$v(x, r, t) = \eta_t(x, t) + \big(r_0 + \eta(x, t)\big)_x u(x, r, t), \quad \text{for} \quad r = r^w(x, t),$$

and expresses the fact that the fluid velocity equals the wall speed $v = r_t^w$. The second boundary condition is actually Newton's second law on the vessel wall written in the form

$$\rho^w h \, \eta_{tt}(x, t) = p^w(x, t) - \frac{E_\sigma h}{r_0^2(x)} \eta(x, t),$$

where ρ^w is the wall density, p^w is the transmural pressure, h is the thickness of the vessel wall, $E_\sigma = E/(1 - \sigma^2)$, where E is the Young's modulus of elasticity with σ denoting the Poisson ratio of the elastic wall. In this study, we assume that E is a constant and in general we will replace in the notation E_σ by E. For more information about the derivation of the Euler equations and the boundary conditions we refer to [3, 19]. Because of the complexity of the Euler's equations, one-dimensional models have been introduce [6, 18], and bidirectional model [1, 12].
In this paper we extend the work [7], and drive some new asymptotic equations of Serre-Green-Naghdi type in cylindrical coordinates, and we drive a new asymptotic one-dimensional model equations of Boussinesq type. The new systems describe inviscid and irrotational fluid flow in elastic vessels of variable diameter and can be used as an alternative case.

II – Asymptotic analysis

II – 1 Non-dimensionalization and normalization

The first step to derive simple mathematical models from the Euler equations is to introduce non-dimensional independent variables that are defined as follows

$$\eta^\star = \frac{\eta}{a}, \quad x^\star = \frac{x}{\lambda}, \quad r^\star = \frac{r}{R}, \quad t^\star = \frac{t}{T}, \quad u^\star = \frac{1}{\epsilon\,\tilde{c}} u,$$

$$v^\star = \frac{1}{\epsilon\,\delta\,\tilde{c}} v, \quad p^\star = \frac{1}{\epsilon\,\rho\,\tilde{c}^2} p,$$

where a is a typical amplitude of the vessel wall displacement, λ is a typical wavelengths of a pulse, R is a vessel's typical radius, $t = \lambda/\tilde{c}$ the characteristic time scale, while $\tilde{c} = \sqrt{Eh/2\rho R}$ is the **Moens-Korteweg** characteristic speed [5]. It is noted that the external pressure is considered zero and neglected. The parameters ϵ and δ characterize the nonlinearity and the dispersion of the system

$$\epsilon = \frac{a}{R}, \quad \delta = \frac{R}{\lambda}.$$

Usually, ϵ and δ are very small. Specifically, we assume that $\epsilon \ll 1$, $\delta^2 \ll 1$, while the Stokes-Ursel number is of order 1 : $\epsilon/\delta^2 = O(1)$. Deleting the $\star$ from the notation below, the non-dimensional form of the Euler becomes [11]

$$u_t + \epsilon u u_x + \epsilon v u_r + p_x = 0, \tag{4}$$

$$\delta^2 (v_t + \epsilon u v_x + \epsilon v v_r) + p_r = 0 \quad \text{for} \quad 0 \leqslant r \leqslant r^w = r_0 + \epsilon\eta, \tag{5}$$

$$r u_x + (r v)_r = 0, \tag{6}$$

$$\delta^2 v_x = u_r, \tag{7}$$

while the boundary conditions are written as

$$v(x, r^w, t) = \eta_t(x, t) + r^w_x u(x, r^w, t), \tag{8}$$

$$p^w(x, t) = p(x, r^w, t) = \alpha\delta^2 \eta_{tt}(x, t) + \beta(x)\eta(x, t) + \delta^2\beta\gamma\eta_t, \tag{9}$$

$$v(x, 0, t) = 0, \tag{10}$$

where

$$\tilde{\alpha} = \frac{\rho^w h}{\rho} \quad \text{and} \quad \tilde{\beta}(x) = \frac{E h}{\rho r_0^2(x)}.$$

Then

$$\alpha = \frac{\tilde{\alpha}}{R} \quad \text{and} \quad \beta(x) = \frac{2R^2\rho}{E h}\tilde{\beta}(x) \quad \text{and} \quad \gamma = \frac{\tilde{\gamma}}{\delta^2 T},$$

where ρ^w is the wall density, h is the thickness of the vessel wall, E is the young modulus of elasticity, R is a vessel's typical radius, $T = \dfrac{\lambda}{c}$ is the characteristic time scale. Equation (7) represents the irrationality of the flow and its is equivalent with the assumption that the flow is potential.

II – 2 Model derivation

It is appropriate to study either the depth averaged velocity or the velocity of the fluid at a certain height above the bottom. In both cases, speed values are expected to be close, because It has been observed that the horizontal velocity of the fluid is usually uniform across the fluid depth. Here, we will obtain approximation models to the Euler equations by using the mean velocity with respect the depth [7], given by

$$\bar{u} = \frac{1}{r_0 + \epsilon\eta} \int_0^{r^w} u(x, r, t)\,\mathrm{d}r. \tag{11}$$

Integration the equation of conservation mass (6), with the kinematic boundary conditions (8), we obtain

$$\int_0^{r^w} r u_x \,\mathrm{d}r + r^w \eta_t(x, t) + r^w r^w_x u(x, r^w, t) = 0. \tag{12}$$

Similarly, integrating the momentum equation (4), we have

$$\int_0^{r^w} u_t \,\mathrm{d}r + \epsilon \int_0^{r^w} u u_x \,\mathrm{d}r + \epsilon \int_0^{r^w} v u_r \,\mathrm{d}r = -\int_0^{r^w} p_x \,\mathrm{d}r. \tag{13}$$

For the derivation of model equations, crucial role plays the assumptions on the pressure field. Using Leibniz rule[1] and the dynamic boundary condition (9) :

$$\int_0^{r^w} p_x \,\mathrm{d}r = \frac{\partial}{\partial x}\int_0^{r^w} p\,\mathrm{d}r - p(x, r^w, t)\, r^w_x = \Big[r^w \bar{p}\Big]_x - \Big(\alpha\,\delta^2 \eta_{tt} + \eta\beta + \delta^2 \beta\gamma\eta_t\Big) r^w_x . \quad (14)$$

To compute $\bar{p}$ we write the r momentum (5) as

$$p_r = -\delta^2\,\Gamma(x, r, t), \quad (15)$$

with $\Gamma(x, r, t) := v_t + \epsilon\, u\, v_x + \epsilon\, v\, v_r$.

Integrating the equation (15) from r to r^w, with the dynamic boundary conditions (9), we obtain

$$\Rightarrow r^w \bar{p} = \Big(\alpha\,\delta^2 \eta_{tt} + \beta\eta + \delta^2 \beta\gamma\eta_t\Big) r^w + \delta^2 \int_0^{r^w}\int_r^{r^w} \Gamma(x, z, t)\,\mathrm{d}z\,\mathrm{d}r .$$

Further, we have

$$\int_0^{r^w} p_x \,\mathrm{d}r = \Big(\alpha\,\delta^2 \eta_{tt} + \beta\eta + \delta^2 \beta\gamma\eta_t\Big)_x r^w + \delta^2 \frac{\partial}{\partial x}\int_0^{r^w}\int_r^{r^w} \Gamma(x, z, t)\,\mathrm{d}z\,\mathrm{d}r. \quad (16)$$

Substituting the equation (16) into (13), we obtain

$$\int_0^{r^w} u_t \,\mathrm{d}r + \epsilon \int_0^{r^w} u\,u_x\,\mathrm{d}r + \Big(\alpha\,\delta^2 \eta_{tt} + \beta\eta + \delta^2 \beta\gamma\eta_t\Big)_x r^w + \delta^2 \frac{\partial}{\partial x}\int_0^{r^w}\int_r^{r^w} \Gamma(x, z, t)\,\mathrm{d}z\,\mathrm{d}r = -\epsilon \int_0^{r^w} v\,u_r\,\mathrm{d}r . \quad (17)$$

Now we compute the Taylor polynomial of u around the bottom 0. Denoting by u_0 the horizontal velocity at the cylindrical axis $r = 0$, we get

$$u(x, r, t) = u_0(x, t) - \frac{1}{4}\,\delta^2 r^2 \frac{\partial^2 u_0}{\partial x^2} + O(\delta^4), \quad (18)$$

$$v(x, r, t) = -\frac{r}{2}\frac{\partial u_0}{\partial x} + O(\delta^2). \quad (19)$$

The integration of (18) yields

$$u_0 = \bar{u} + \frac{1}{12}\,\delta^2 (r^w)^2 \frac{\partial^2 \bar{u}}{\partial x^2} + O(\delta^4), \quad (20)$$

and therefore, (18) becomes

$$u(x, r, t) = \bar{u} + \frac{1}{12}\,\delta^2 (r^w)^2 \frac{\partial^2 \bar{u}}{\partial x^2} - \frac{1}{4}\,\delta^2 r^2 \frac{\partial^2 \bar{u}}{\partial x^2} + O(\delta^4). \quad (21)$$

1. $\frac{\partial}{\partial x}\left(\int_{a(t)}^{b(t)} f(x, t)\,\mathrm{d}x\right) = \int_{a(t)}^{b(t)} f_t(x, t)\,\mathrm{d}x + f(b(t), t).b'(t) - f(a(t), t).a'(t).$

Moreover, the vertical velocity is given by

$$v(x, r, t) = -\frac{r}{2}\frac{\partial \bar{u}}{\partial x} + O(\delta^2). \tag{22}$$

Using equation (21) to evaluate the following integrals

$$\int_0^{r^w} u\, u_x \,\mathrm{d}r = r^w \bar{u}\,\bar{u}_x + \frac{1}{6}\delta^2 r_x^w (r^w)^2 \bar{u}\,\bar{u}_{xx} + O(\delta^4), \tag{23}$$

$$\int_0^{r^w} u_t \,\mathrm{d}r = r^w \bar{u}_t + \frac{1}{6}\delta^2 r_t^w (r^w)^2 \bar{u}_{xx}, \tag{24}$$

$$\int_0^{r^w} v\, u_r \,\mathrm{d}r = \delta^2 \int_0^{r^w} \frac{r^2}{4}\bar{u}_x \bar{u}_{xx} \,\mathrm{d}r = \frac{1}{12}\delta^2 \bar{u}_x \bar{u}_{xx} (r^w)^3 + O(\delta^4). \tag{25}$$

Substituting equation (22) into $\Gamma(x, r, t)$, allows us to rewrite $\Gamma(x, r, t)$

$$\Gamma(x, r, t) = -\frac{r}{2}\left[\bar{u}_{xt} + \epsilon \bar{u}\,\bar{u}_{xx} - \frac{\epsilon}{2}(\bar{u}_x)^2\right] + O(\delta^2, \epsilon\delta^2). \tag{26}$$

Combining (26), (25), (24), (23) and (17) and taking $\bar{u}_{xt} + \epsilon \bar{u}\,\bar{u}_{xx} - \epsilon(\bar{u}_x)^2$ is independent of r, we have

$$\begin{aligned} r^w \bar{u}_t + \frac{\delta^2}{6} r_t^w (r^w)^2 \bar{u}_{xx} + \epsilon r^w \bar{u}\,\bar{u}_x + \frac{\epsilon\delta^2}{6} r_x^w (r^w)^2 \bar{u}\,\bar{u}_{xx} + \big(\alpha\delta^2 \eta_{tt} + \beta\eta + \\ \delta^2\beta\gamma\eta_t\big)_x r^w + \frac{\epsilon\delta^2}{12}(r^w)^3 \bar{u}_x \bar{u}_{xx} - \frac{\delta^2}{6}\frac{\partial}{\partial x}\big[(r^w)^3(\bar{u}_{xt} + \epsilon\bar{u}\,\bar{u}_{xx} \\ - \frac{\epsilon}{2}(\bar{u}_x)^2\big] = O(\delta^4). \end{aligned} \tag{27}$$

If we return to the equation (12), and substitute (21) in the first term, and assuming $\delta \ll 1$ and $\epsilon = O(1)$, we get

$$\frac{(r^w)^2}{2}\bar{u}_x + \frac{\delta^2}{12} r_x^w (r^w)^3 \bar{u}_{xx} - \frac{\delta^2}{48}(r^w)^4 \bar{u}_{xxx} + r^w \eta_t + r^w r_x^w u^w(x, t) = O(\delta^4), \tag{28}$$

$$\begin{aligned} r^w \bar{u}_t + \frac{\delta^2}{6} r_t^w (r^w)^2 \bar{u}_{xx} + \epsilon r^w \bar{u}\,\bar{u}_x + \frac{\epsilon\delta^2}{6} r_x^w (r^w)^2 \bar{u}\,\bar{u}_{xx} + \big(\alpha\delta^2 \eta_{tt} + \beta\eta \\ + \delta^2\beta\gamma\eta_t\big)_x r^w + \frac{\epsilon\delta^2}{12}(r^w)^3 \bar{u}_x \bar{u}_{xx} - \frac{\delta^2}{6}\frac{\partial}{\partial x}\left[(r^w)^3(\bar{u}_{xt} + \epsilon\bar{u}\,\bar{u}_{xx} - \frac{\epsilon}{2}(\bar{u}_x)^2\right] \\ = O(\delta^4). \end{aligned} \tag{29}$$

The integration of equation (7) from r to r^w and subsequent solution with respect u yields

$$u(x, r, t) = u(x, r^w, t) - \delta^2 \int_r^{r^w} v_x(x, s, t)\,\mathrm{d}s. \tag{30}$$

We continue with the derivation of analogues asymptotic approximations for the radial velocity v. We consider the function [10]

$$Q(x, r, t) = \frac{1}{r}\int_0^r s\, u(x, s, t)\,\mathrm{d}s. \tag{31}$$

Using (30), we obtain

$$Q(x, r, t) = \frac{1}{r} \int_0^r s\, u^w(x, t)\, \mathrm{d}s + O(\delta^2) = \frac{r}{2} u^w(x, t) + O(\delta^2). \tag{32}$$

The integration of the equation (6) by parts yields

$$v(x, r, t) = -\frac{1}{r} \int_0^{r^w} s\, u_x\, \mathrm{d}s = -Q_x(x, r, t), \tag{33}$$

which simplifies to

$$v(x, r, t) = -\frac{r}{2} u_x^w(x, t) + O(\delta^2), \tag{34}$$

if we combine (34) with (30), which leads to the high-order approximation of the horizontal velocity, we obtain

$$u^w(x, t) = u(x, r, t) - \delta^2 u_{xx}^w(x, t) \frac{(r^w)^2 - r^2}{4} + O(\delta^4). \tag{35}$$

Using the definition of u^w, we consequently have

$$u^w(x, t) = u - \delta^2 u_{xx}(x, r, t) \frac{(r^w)^2 - r^2}{4} + O(\delta^4), \tag{36}$$

with $u(x, r, t) = \bar{u}(x, t) + \frac{1}{12}\delta^2 (r^w)^2 \bar{u}_{xx}(x, t) - \frac{1}{4}\delta^2 r^2 \bar{u}_{xx}(x, t) + O(\delta^4)$. A short calculation yields

$$u^w(x, t) = \bar{u} - \frac{1}{6}\delta^2 (r^w)^2 \bar{u}_{xx} + O(\delta^4). \tag{37}$$

Using the above computations and returning to the main system (28) – (29), we obtain the PDEs

$$\frac{(r^w)^2}{2}\bar{u}_x - \frac{\delta^2}{12} r_x^w (r^w)^3 \bar{u}_{xx} - \frac{\delta^2}{48}(r^w)^4 \bar{u}_{xxx} + r^w \eta_t + r^w r_x^w \bar{u} = O(\delta^4), \tag{38}$$

$$r^w \bar{u}_t + \frac{\delta^2}{6} r_t^w (r^w)^2 \bar{u}_{xx} + \epsilon r^w \bar{u}\bar{u}_x + \frac{\epsilon\delta^2}{6} r_x^w (r^w)^2 \bar{u}\bar{u}_{xx} + \big(\alpha\delta^2 \eta_{tt} + \beta\eta$$

$$+ \delta^2 \beta\gamma\eta_t\big)_x r^w + \frac{\epsilon\delta^2}{12}(r^w)^3 \bar{u}_x \bar{u}_{xx} - \frac{\delta^2}{6}\frac{\partial}{\partial x}\left[(r^w)^3\left(\bar{u}_{xt} + \epsilon\bar{u}\bar{u}_{xx} - \frac{\epsilon}{2}(\bar{u}_x)^2\right)\right]$$

$$= O(\delta^4). \tag{39}$$

Here, as before, $r^w \equiv r_0(x) + \epsilon\eta(x, t)$ with $r_0 = \text{const}$. Finally we arrive at the following PDE system with variable coefficients

$$\frac{r_0}{2}\bar{u}_x + \epsilon\frac{\eta}{2}\bar{u}_x - \frac{\epsilon\delta^2}{12}\eta_x (r_0 + \eta)^2 \bar{u}_{xx} - \frac{\delta^2}{48}(r_0 + \epsilon\eta)^3 \bar{u}_{xxx} + \eta_t + \epsilon\eta_x \bar{u} = O(\delta^4), \tag{40}$$

$$\bar{u}_x + \epsilon\,\bar{u}\bar{u}_x + \frac{\epsilon\,\delta^2}{6}\eta_t\,(r_0 + \epsilon\,\eta)\,\bar{u}_{xx} + \frac{\epsilon^2\,\delta^2}{6}\eta_x\,(r_0 + \epsilon\,\eta)\,\bar{u}\,\bar{u}_{xx} + \Bigg(\alpha\,\delta^2\,\eta_{xx} + \beta\,\eta + \delta^2\beta\,\gamma\,\eta_t\Bigg)_x + \frac{\epsilon\,\delta^2}{16}(r_0 + \epsilon\,\eta)^2\,\bar{u}_x\bar{u}_{xx} - \frac{\delta^2}{6\,(r_0 + \epsilon\,\eta)}\frac{\partial}{\partial x}\Bigg[(r_0 + \epsilon\,\eta)^3\,(\bar{u}_{xt} + \epsilon\,\bar{u}\,\bar{u}_{xx} - \frac{\epsilon}{2}\,(\bar{u}_x)^2)\Bigg] = O(\delta^4). \quad (41)$$

III – Model reduction

III – 1 The classical cylindrical Boussinesq system

Considering long waves of small amplitude, i.e., when $\delta \ll 1$ and $\epsilon \ll 1$ the system could be simplified further. For example, keeping the terms of $O(\epsilon, \delta^2)$, with $r_0 =$ const, we obtain a constant-coefficient Boussinesq-type nonlinear system in dimensional variables for general constant radius $r_0 > 0$:

$$\frac{r_0}{2}\,\bar{u}_x + \frac{\eta}{2}\,\bar{u}_x + \eta_t - \frac{r_0^3}{48}\,\bar{u}_{xxx} + \eta_x\,\bar{u} = 0\,, \quad (42)$$

$$\bar{u}_t + \bar{u}\,\bar{u}_x + \tilde{\alpha}\,\eta_{xtt} + \tilde{\beta}\,\eta_x + \tilde{\beta}\,\tilde{\gamma}\,\eta_{xt} - \frac{r_0^2}{6}\,\bar{u}_{xxt} = 0. \quad (43)$$

III – 2 A unidirection model : Viscous KdV-and BBM-type equations

In order to drive such models we consider the following dimensionless variables :

$$\eta^\star = \frac{\eta}{a}, \quad x^\star = \frac{x}{\lambda}, \quad r^\star = \frac{r}{r_0}, \quad t^\star = \frac{t}{T}, \quad u^\star = \frac{u}{c_0}, \quad (44)$$

where here $c_0 = \dfrac{a}{r_0}\sqrt{\dfrac{2\,E\,h}{\rho\,r_0}}$ is a modified Moens-Korteweg characteristic speed and $T = 2\,\dfrac{a\,\lambda}{r_0\,c_0}$. The system (42) – (43) in dimensionless variables then is written as

$$\eta^\star_{t^\star} + u^\star_{x^\star} + \varepsilon\,\eta^\star\,u^\star_{x^\star} + 2\,\varepsilon\,\eta^\star_{x^\star}\,u^\star - \frac{\delta^2}{24}\,u^\star_{x^{\star\star\star}} = 0\,, \quad (45)$$

$$u^\star_{t^\star} + \eta^\star_{x^\star} + 2\,\varepsilon\,u^\star\,u^\star_{x^\star} + \frac{1}{2}\,\delta^2\,\alpha\,\eta^\star_{x^\star t^\star t^\star} + \gamma^\star\,\delta^2\,\eta^\star_{x^\star\,t^\star} - \frac{1}{6}\,\delta^2\,u^\star_{x^\star x^\star t^\star} = 0, \quad (46)$$

where $\gamma^\star = \dfrac{1}{2}\dfrac{\sigma}{\varepsilon\,\delta^2}\,\gamma$, $\sigma = \dfrac{c_0}{\lambda}$, and $\varepsilon = \dfrac{a}{r_0}$, $\delta = \dfrac{r_0}{\lambda}$. From (45), we observe that $\eta^\star_{t^\star} = -\,u^\star_{x^\star} + O(\varepsilon, \delta^2)$, and thus $\eta^\star_{x^\star t^\star t^\star} = -\,u^\star_{x^\star x^\star t^\star} + O(\varepsilon, \delta^2)$ and $\eta^\star_{x^\star t^\star} = -u^\star_{x^\star x^\star} + O(\varepsilon, \delta^2)$. Considering a flow mainly towards in the direction of x, we can use the low-order approximation for unidirectional wave propagation [17]

$$\bar{u^\star} = \eta^\star + \varepsilon\,A + \delta^2\,B + O(\varepsilon^2, \delta^4, \varepsilon\,\delta^2)\,,$$

where A and B are two unknown function of $x^\star$ and $t^\star$. Hence in dimensional variables the viscous KdV equation takes the form

$$\eta_t + \tilde{c}\eta_x + \frac{5}{2}\frac{\tilde{c}}{r_0}\eta\eta_x + \frac{\tilde{c}(12\tilde{\alpha} + 3r_0)r_0}{48}\eta_{xxx} - \frac{r_0}{4}\tilde{\beta}\tilde{\gamma}\eta_{xx} = 0. \qquad (47)$$

In dimensional variables the viscous BBM equation takes the form

$$\eta_t + \tilde{c}\eta_x + \frac{5}{2}\frac{\tilde{c}}{r_0}\eta\eta_x - \frac{(12\tilde{\alpha} + 3r_0)r_0}{48}\eta_{xxt} - \frac{r_0}{4}\tilde{\gamma}\tilde{\beta}\eta_{xx} = 0, \qquad (48)$$

$$\tilde{c} = \sqrt{\frac{Eh}{2\rho r_0}}.$$

In the absence of any form of dissipation, and after taking the inviscid limit $\tilde{\gamma} \to 0$, suppose $\eta(x, t) = \eta(x - c_s t) = \eta(\zeta)$, it is known that the viscous KdV equation possesses classical solitary waves propagating with speed c_s, given by the formula

$$\eta(x, t) = \frac{48(c_s - \tilde{c})r_0}{40\tilde{c}}\operatorname{sech}^2\left(\sqrt{\frac{12(c_s - \tilde{c})}{\tilde{c}(12\tilde{\alpha} + 3r_0)r_0}}(x - c_s t)\right).$$

For the viscous BBM equation (48), classical solitary waves propagating with speed c_s can be computed explicitly and have the form

$$\eta(x, t) = \frac{48(c_s - \tilde{c})r_0}{40\tilde{c}}\operatorname{sech}^2\left(\sqrt{\frac{12(c_s - \tilde{c})}{c_s(12\tilde{\alpha} + 3r_0)r_0}}(x - c_s t)\right).$$

The dispersion relation $\omega = \omega(k)$ of the viscous KdV equation (47) can be easily computed

$$\omega = \tilde{c}k - \frac{r_0\tilde{c}(12\tilde{\alpha} + 3r_0)k^3}{48} - \frac{\mathrm{i}}{4}r_0\gamma\tilde{\beta}k^2.$$

Similarly, the dispersion relation of the viscous derived BBM equation (48) is given by

$$\omega = \frac{48\tilde{c}k}{48 + r_0(12\tilde{\alpha} + 3r_0)k^2} - \mathrm{i}\frac{12\gamma r_0\tilde{\beta}k^2}{48 + r_0(12\tilde{\alpha} + 3r_0)k^2}.$$

III − 3 Conservation laws and symmetries

In the general case, when the unperturbed radius of the vessel is variable, that is, $r_0 = r_0(x)$ and $r^w = r_0(x) + \eta(x, t)$, the system (38), (39) can be written in dimensional variables as

$$\frac{(r^w)^2}{2}\bar{u}_x - \frac{1}{12}r^w_x(r^w)^3\bar{u}_{xx} - \frac{1}{48}(r^w)^4\bar{u}_{xxx} + r^w\eta_t(x, t) + r^w r^w_x\bar{u} = 0, \quad (49)$$

$$r^w\bar{u}_t + \frac{1}{6}r^w_t(r^w)^2\bar{u}_{xx} + r^w\bar{u}\bar{u}_x + \frac{1}{6}r^w_x(r^w)^2\bar{u}\bar{u}_{xx} + \Big(\tilde{\alpha}\eta_{tt}(x, t) + \tilde{\beta}\eta(x, t)$$

$$+ \tilde{\beta}\gamma\eta_t\Big)_x r^w + \frac{1}{12}(r^w)^3\bar{u}_x\bar{u}_{xx} - \frac{1}{6}\frac{\partial}{\partial x}\left[(r^w)^3\left(\bar{u}_{xt} + \bar{u}\bar{u}_{xx} - \frac{1}{2}(\bar{u}_x)^2\right)\right] = 0. \qquad (50)$$

The PDE system (49) – (50) admits only one conservation law correspond to the set of multipliers : $\lambda_1(t, x, \eta, \bar{u}, \eta_x, \bar{u}_x, \eta_{xx}, \bar{u}_{xx}, \eta_{xxx}, \bar{u}_{xxx}) = 1$, $\lambda_2(t, x, \eta, \bar{u}, \eta_x, \bar{u}_x, \eta_{xx}, \bar{u}_{xx}, \eta_{xxx}, \bar{u}_{xxx}) = 0$, which is given by

$$\left[\eta r_0 + \frac{1}{2}\eta^2\right]_t + \left[-\frac{1}{48}\eta^4\bar{u}_{xx} - \frac{1}{12}\eta^3 r_0\bar{u}_{xx} - \frac{1}{8}\eta^2 r_0^2\bar{u}_{xx} + \frac{1}{2}\eta^2\bar{u} - \frac{1}{48}r_0^4\bar{u}_{xx}\right.$$
$$\left. + r_0\eta\bar{u} + \frac{1}{2}r_0^2\bar{u} - \frac{1}{12}\eta r_0^3\bar{u}_{xx}\right]_x = 0\,.$$

Let us compute also the symmetry group of the equations (38) – (39), we find 7 cases to study.

- Case 1 : This is the most general one (for any α, β, γ and $r_0(x)$). IN this case, the system admits a single time-translation symmetry
  $X_1 = \partial_t$.
- Case 2 : In this case, $r_0'' = 0 \Rightarrow r_0(x) = r_{00} + x\,.\,r_{01}$, and one has two symmetries
  $X_1 = \partial_t$,
  $X_2 = \partial_x - r_{01}\,\partial_\eta$.
- Case 3 : for $\beta = 0$, one also has two symmetries that are given by
  $X_1 = \partial_t$,
  $X_2 = \bar{u}\,\partial_{\bar{u}} - t\,\partial_t$.
- Case 4 : for $\beta = 0$ and $r'' = 0$, we obtain three symmetries
  $X_1 = \partial_t$,
  $X_2 = \partial_x - r_{01}\,\partial_\eta$,
  $X_3 = \bar{u}\,\partial_{\bar{u}} - t\,\partial_t$.
- Case 5 : In this case, $r' = 0 \Rightarrow r_0 = r_{00}$, we have two symmetries
  $X_1 = \partial_x$,
  $X_2 = \partial_t$.
- Case 6 : In this case, $\alpha = \gamma = r' = 0$, we get three-dimensional symmetries algebra spanned by the generators
  $X_1 = \partial_x$,
  $X_2 = \partial_t$,
  $X_3 = t\,\partial_x + \partial_{\bar{u}}$.
- Case 7 : For $\alpha = \beta = r' = 0$, we get four symmetries follows as
  $X_1 = \partial_x$,
  $X_2 = \partial_t$,
  $X_3 = t\,\partial_x + \partial_{\bar{u}}$,
  $X_4 = \bar{u}\,\partial_{\bar{u}} - t\,\partial_t$.

IV – Conclusions and perspectives

In this paper we derived new weakly dispersive systems derived that model inviscid fluid flow in viscoelastic vessels. Moreover we obtained a fully nonlinear cylindrical analog of Serre equations, and derived unidirectional equations of BBM and KdV type to the Boussinesq-type system. The numerical analysis of derived models is left for future research.

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