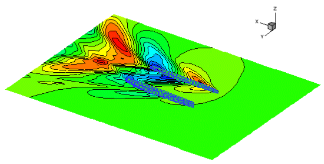

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GENERALIZED BOUNDARY-INTEGRAL REPRESENTATION OF 3D FLOW ABOUT A SHIP ADVANCING IN REGULAR WAVES

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Résumé

Cette étude porte sur le problème fondamental qui consiste à déterminer l'écoulement potentiel correspondant à un écoulement donné à la surface frontière du domaine de l'écoulement. Une nouvelle représentation du potentiel est obtenue. Cette représentation du potentiel généralise deux représentations connues : une représentation classique qui exprime le potentiel Á au moyen d'une fonction de Green G et de son gradient r G, et la représentation faiblement singulière obtenue dans [1], qui définit Á au moyen de G et d'une fonction de Green vectorielle G comparable à G (en particulier, la fonction G n'est pas plus singulière que G). En fait, ces deux représentations sont complémentaires et correspondent à des cas particuliers de la représentation généralisée obtenue dans cette étude. La représentation généralisée du potentiel est appliquée ` a la diffraction-radiation par un navire avançant dans une houle régulière (et les cas particuliers d'un écoulement permanent autour d'un navire avançant en eau calme et de la diffraction-radiation sans vitesse d'avance), ainsi qu'aux cas limites correspondant à une gravité nulle ou infinie. Pour la diffraction-radiation avec vitesse d'avance, on considère deux fonctions de Green associées à la condition de surface libre linéarisée : la fonction de Green usuelle, qui satisfait la condition à la surface libre partout, et la fonction de Green plus simple obtenue dans [2] .

Summary

The fundamental problem of determining the potential flow that corresponds to a given flow at the boundary surface of the flow domain is considered. A generalized boundary-integral representation of the potential is given. This potential representation is an extension of two alternative basic representations of the potential : a classical representation, which defines a velocity potential Á in terms of a Green function G and its gradient r G, and the alternative weakly-singular potential representation given in [1], which defines Á in terms of G and a related vector Green function G that is comparable to G (in particular, G is no more singular than G). In fact, these two alternative basic potential representations are complementary, and are special cases of the generalized representation given in the present study. The generalized potential representation is applied to free-surface flows in the infinite-gravity and zero-gravity limits, and wave diffraction-radiation by a ship advancing in time-harmonic waves (and the special cases corresponding to diffraction-radiation without forward speed and steady flow about a ship advancing in calm water). For wave diffraction-radiation with forward speed, two alternative Green functions associated with the linearized free-surface boundary condition are considered : the usual Green function, which satisfies the free-surface condition everywhere, and the simpler farfield Green function given in [2].

[1] Weakly-singular boundary-integral representations of free-surface flows about ships or offshore structures, Journal of Ship Research, 2004, 48:31-44

[2] A simple Green function for diffraction-radiation of time-harmonic waves with forward speed, Ship Technology Re-search (Schiffstechnik), 2004, 51:35-52.

1. Introduction

Wave diffraction-radiation by a ship advancing through regular (time-harmonic) waves at the free surface of a large body of water of uniform depth D is considered within the framework of a 3D potential-flow frequency-domain analysis. This basic core issue is one of the most classical and important problem in ship hydrodynamics. Indeed, 3D wave diffraction-radiation with forward speed (in deep water or in finite water depth) is relevant to hydrodynamic hull-form design and optimization (notably of fast and unconventional vessels and at early stages), viscous ship hydrodynamics (via coupling with RANSE nearfield calculation methods), and ship motions in large waves (added-mass and wave-damping coefficients can be used for effective nonlinear time-domain simulations). Accordingly, the problem has been extensively considered in the literature, with limited success however due to major difficulties related to forward speed effects. This literature, the basic mathematical and numerical difficulties of the problem, and its considerable practical importance are reviewed in *Noblesse and Yang (2004a,b,c)*.

The z axis is vertical and points upward, and the mean free surface is taken as the plane $z=0$. For steady and time-harmonic flow about a ship advancing in calm water or in waves, the x axis is chosen along the path of the ship and points toward the bow. Coordinates are nondimensional with respect to a reference length L, e.g. the ship length. The fluid velocity is nondimensional with respect to a reference velocity U, e.g. $U=\sqrt{gL}$ (where g is the acceleration of gravity) or $U=\mathcal{U}$ (the ship speed), and the velocity potential ϕ is nondimensional with respect to the reference potential UL.

Let Σ_B be a surface located outside the viscous boundary layer that surrounds the ship hull. The surface Σ_B includes the outer edge of the viscous wake trailing the ship, or a surface outside the viscous wake. If viscous effects are ignored, Σ_B may be taken as the mean wetted ship hull. For a ship equipped with lifting surfaces, e.g. a sailboat, Σ_B also includes the two sides of every vortex sheet behind the ship hull. For a multihull ship, the hull+wake surface Σ_B consists of several component surfaces, which correspond to the separate hull components of the ship and their wakes.

The flow domain is bounded by the surface

$$\Sigma = \Sigma_B \cup \Sigma_0 \cup \Sigma_D \tag{1}$$

where Σ_0 is the portion of the mean free-surface plane $z=0$ located outside the "body" surface Σ_B, and Σ_D is the sea floor $z=-D/L$, assumed to be a rigid wall. Let Γ represent the intersection curve between the surfaces Σ_B and Σ_0, i.e. the intersection curve of the body surface Σ_B with the free-surface plane. The unit vector $\mathbf{n} = (n^x, n^y, n^z)$ is normal to the boundary surface Σ and points into the flow domain. Thus, $\mathbf{n} = (0, 0, -1)$ at the free surface Σ_0 and $\mathbf{n} = (0, 0, 1)$ at the sea floor Σ_D. The unit vector $\mathbf{t} = (t^x, t^y, 0)$ is tangent to the boundary curve Γ and oriented clockwise (looking down). Finally, the unit vector $\mathbf{n}^\Gamma = (-t^y, t^x, 0)$ is normal to the curve Γ in the free-surface plane $z=0$ and points into the flow domain (like the unit vector $\mathbf{n}$ normal to the boundary surface Σ).

Let $\widetilde{\mathbf{x}} = (\widetilde{x}, \widetilde{y}, \widetilde{z})$ and $\mathbf{x} = (x, y, z)$ stand for a field point and a singularity point, respectively, associated with a Green function $G(\widetilde{\mathbf{x}};\mathbf{x})$. The field point $\widetilde{\mathbf{x}}$ lies inside the flow domain, and the singularity point $\mathbf{x}$ is located on the boundary surface Σ. Furthermore, X and Y are defined as

$$X = \widetilde{x} - x \qquad Y = \widetilde{y} - y \tag{2}$$

Hereafter, $\widetilde{\phi}$ stands for the velocity potential at a field point $\widetilde{\mathbf{x}}$, and ϕ and $\nabla\phi$ represent the potential and the velocity at a boundary point $\mathbf{x}$. Furthermore, $\widetilde{\nabla}$ and ∇ stand for the differential operators $\widetilde{\nabla} = (\partial_{\widetilde{x}}, \partial_{\widetilde{y}}, \partial_{\widetilde{z}})$ and $\nabla = (\partial_x, \partial_y, \partial_z)$.

2. Classical potential-flow representation

The potential $\widetilde{\phi} = \phi(\widetilde{\mathbf{x}})$ at a field point $\widetilde{\mathbf{x}}$ within a flow domain bounded by a closed boundary surface Σ is defined in terms of the boundary values of the potential ϕ and its normal derivative $\mathbf{n}\cdot\nabla\phi$ by the classical boundary-integral representation

$$\widetilde{\phi} = \int_\Sigma d\mathcal{A}\,(G\,\mathbf{n}\cdot\nabla\phi - \phi\,\mathbf{n}\cdot\nabla G) \tag{3}$$

where $d\mathcal{A}$ stands for the differential element of area at a point $\mathbf{x}$ of the boundary surface Σ. The representation (3) defines the potential in terms of boundary distributions of sources (with strength

$\mathbf{n}\cdot\nabla\phi$) and normal dipoles (strength ϕ), and involves a Green function G and the first derivatives of G . Differentiation of the potential representation (3) yields

$$\widetilde{\nabla}\widetilde{\phi} = \int_{\Sigma} d\mathcal{A}\,[\,(\mathbf{n}\cdot\nabla\phi)\,\widetilde{\nabla}G - \phi\,\widetilde{\nabla}(\mathbf{n}\cdot\nabla G)\,] \tag{4}$$

This classical velocity representation involves second derivatives of G . The potential representation (3) holds for a field point $\widetilde{\mathbf{x}}$ inside the flow domain, strictly outside the boundary surface Σ . This restriction stems from the well-known property that the potential defined by the dipole distribution in (3) is not continuous at the surface Σ . Indeed, $\widetilde{\phi}$ on the left of (3) becomes $\widetilde{\phi}/2$ at a point $\widetilde{\mathbf{x}}$ of the boundary surface Σ (if Σ is smooth at $\widetilde{\mathbf{x}}$).

3. Weakly-singular potential-flow representation

An alternative boundary-integral representation is given in *Noblesse and Yang (2004a).* This alternative representation is obtained using a vector Green function $\mathbf{G}$ associated with the scalar Green function G in (3) via the relation

$$\nabla\times\mathbf{G} = \nabla G \tag{5}$$

The relation (5) implies that $\mathbf{G}$ is no more singular than G . The relation (5) between a scalar Green function G and a vector Green function $\mathbf{G}$ is analogous to the relation $\nabla\times\mathbf{\Psi} = \nabla\phi$ between a velocity potential ϕ and a stream function $\mathbf{\Psi}$. The relation (5) does not define a unique vector Green function $\mathbf{G}$; indeed, if $\mathbf{G}$ satisfies (5), $\mathbf{G}+\nabla H$ also satisfies (5) for an an arbitrary scalar function H .

The identity $\nabla\times(\,\phi\,\mathbf{G}\,) = \phi\,\nabla\times\mathbf{G} + \nabla\phi\times\mathbf{G}$ and (5) yield

$$[\,\nabla\times(\,\phi\,\mathbf{G}\,)\,]\cdot\mathbf{n} = \phi\,\nabla G\cdot\mathbf{n} + (\,\nabla\phi\times\mathbf{G}\,)\cdot\mathbf{n}$$

Integration of this identity over a closed boundary surface Σ then yields

$$-\int_{\Sigma} d\mathcal{A}\;\phi\,\mathbf{n}\cdot\nabla G = \int_{\Sigma} d\mathcal{A}\,(\mathbf{n}\times\nabla\phi)\cdot\mathbf{G} \tag{6}$$

The foregoing identities, with $\mathbf{G}$ taken as ∇H , yield

$$0 = \int_{\Sigma} d\mathcal{A}\,(\mathbf{n}\times\nabla\phi)\cdot\nabla H \tag{7}$$

The field point $\widetilde{\mathbf{x}}$ in (6) is inside the flow domain, strictly outside the boundary surface Σ . The transformation (6) expresses a surface integral involving the potential ϕ and the derivative ∇G of a Green function G as an integral that involves $\nabla\phi$ and the vector Green function $\mathbf{G}$, which is comparable to G . Thus, the transformation (6) corresponds to an integration by parts $(\,\phi\,,\nabla G) \longrightarrow (\nabla\phi, G)$.

Substitution of the transformation (6) into the classical potential representation (3) yields

$$\widetilde{\phi} = \int_{\Sigma} d\mathcal{A}\,[\,G\,\mathbf{n}\cdot\nabla\phi + \mathbf{G}\cdot(\mathbf{n}\times\nabla\phi)\,] \tag{8}$$

The identity (7) shows that expression (8) defines a unique potential $\widetilde{\phi}$, even though the relation (5) does not define a unique vector Green function $\mathbf{G}$. The alternative potential representation (8) involves a Green function G and the related vector Green function $\mathbf{G}$, which is comparable to (in particular, is no more singular than) G as already noted. Thus the potential representation (8) is weakly singular in comparison to the classical representation (3), which involves G and ∇G . The potential $\widetilde{\phi}$ defined by the weakly-singular potential representation (8) is continuous at the boundary surface Σ , whereas the classical boundary-integral representation (3) does not define a potential $\widetilde{\phi}$ that is continuous at Σ .

The well-known velocity representation

$$\widetilde{\nabla}\widetilde{\phi} = \int_{\Sigma} d\mathcal{A}\,[\,(\mathbf{n}\cdot\nabla\phi)\,\widetilde{\nabla}G + (\mathbf{n}\times\nabla\phi)\times\widetilde{\nabla}G\,] \tag{9}$$

can be obtained via judicious differentiation of the weakly-singular potential representation (8), in the manner shown in *Noblesse and Yang (2002).* The velocity representation (9) only involves first derivatives of G and thus is weakly singular in comparison to the velocity representation (4), which involves second derivatives of G. The velocity representation (9) is applied to wave diffraction-radiation of time-harmonic waves by a ship or an offshore structure in *Noblesse (2001)*. A drawback of the velocity representation (9) and the related representations given in *Noblesse (2001)* is that they do not necessarily define a potential flow; see e.g. *Hunt (1980).* Indeed, the velocity representation (9) is not identical to the gradient of the potential representation (8), as shown in *Noblesse and Yang (2002).*

A specific vector Green function that satisfies (5) is

$$\mathbf{G} = (\,G_y^z\,, -G_x^z\,, 0\,) \tag{10}$$

Here, a subscript or superscript attached to G indicates differentiation or integration, respectively. The vector Green function (10) is used here, as in *Noblesse and Yang (2004a).* Alternative vector Green functions can be used; e.g. the vector Green functions $\mathbf{G} = (\,0, G_z^x\,, -G_y^x\,)$ and $\mathbf{G} = (-G_z^y\,, 0, G_x^y\,)$ are considered in *Noblesse and Yang (2002).*

4. Generalized potential representation

The weakly-singular boundary-integral representation (8) and the classical representation (3) can be regarded as special cases of a more general family of potential representations, as now shown. The basic potential representation (3) can be expressed as

$$\widetilde{\phi} = \int_\Sigma d\mathcal{A}\,[\,G\,\mathbf{n}\cdot\nabla\phi - (1-P)\,\phi\,\mathbf{n}\cdot\nabla G - P\,\phi\,\mathbf{n}\cdot\nabla G\,] \tag{11a}$$

where $P = P(\mathbf{x}\,;\widetilde{\mathbf{x}})$ stands for a function of $\mathbf{x}$ and $\widetilde{\mathbf{x}}$. The transformation (6), with ϕ replaced by $P\phi$, yields

$$-\int_\Sigma d\mathcal{A}\,P\phi\,\mathbf{n}\cdot\nabla G = \int_\Sigma d\mathcal{A}\,\mathbf{G}\cdot[\,\mathbf{n}\times\nabla(P\phi)\,] \tag{11b}$$

The potential representation (11a) and the transformation (11b) yield

$$\widetilde{\phi} = \int_\Sigma d\mathcal{A}\,[\,G\,\mathbf{n}\cdot\nabla\phi - (1-P)\,\phi\,\mathbf{n}\cdot\nabla G + P\,\mathbf{G}\cdot(\mathbf{n}\times\nabla\phi) + \phi\,\mathbf{G}\cdot(\mathbf{n}\times\nabla P\,)\,] \tag{12}$$

The potential representation (12) generalizes the classical representation (3) and the weakly-singular representation (8), which correspond to the special cases $P = 0$ and $P = 1$, respectively.

5. Potential representations for free-space Green function

The potential representation (12) is now considered for the simplest case when the Green function is chosen as the fundamental free-space Green function, defined as $4\pi G = -1/r$ with

$$r = \sqrt{\mathbf{X}\cdot\mathbf{X}} \qquad \mathbf{X} = (\,X, Y, Z\,) \qquad Z = \widetilde{z} - z \tag{13}$$

X and Y are given by (2). Expressions (12) and (13) yield

$$\widetilde{\phi} = \frac{-1}{4\pi}\int_\Sigma d\mathcal{A}\left(\frac{\mathbf{n}\cdot\nabla\phi}{r} - \phi\,\frac{1-P}{r^2}\,\frac{\mathbf{X}\cdot\mathbf{n}}{r} + P\,\mathbf{S}\cdot(\mathbf{n}\times\nabla\phi) + \phi\,\mathbf{S}\cdot(\mathbf{n}\times\nabla P)\right) \tag{14}$$

where $\mathbf{S}$ satisfies the relation $\nabla\times\mathbf{S} = \nabla(1/r)$ and is chosen as $\mathbf{S} = [\,(1/r)_y^z\,, -(1/r)_x^z\,, 0\,]$ in accordance with (10). The function $(1/r)^z$ and its derivatives with respect to x and y are given by

$$(1/r)^z = -\,\mathrm{sign}(Z)\,\ln(\,r + |\,Z\,|\,) \qquad \left\{\begin{matrix}(1/r)_x^z\\(1/r)_y^z\end{matrix}\right\} = \frac{\mathrm{sign}(Z)}{r + |\,Z\,|}\left\{\begin{matrix}X/r\\Y/r\end{matrix}\right\} \tag{15}$$

Thus, we have $\mathbf{S} = \mathbf{s}/r$ where $\mathbf{s}$ is given by

$$\mathbf{s} = \frac{\mathrm{sign}(d^{\,z})}{1+|d^{\,z}|}\,(d^{\,y}, -d^{\,x}, 0) \qquad (d^{\,x}, d^{\,y}, d^{\,z}) = \mathbf{d} = \frac{(x-\widetilde{x}, y-\widetilde{y}, z-\widetilde{z})}{r} \tag{16}$$

This definition of $\mathbf{d}$ yields $\mathbf{d}=-\mathbf{X}/r$, $\nabla r=\mathbf{d}$ and $\nabla(1/r)=-\,\mathbf{d}/r^2$. Thus, (14) becomes

$$\widetilde{\phi}=\frac{-1}{4\pi}\int_\Sigma \frac{d\mathcal{A}}{r}\left(\mathbf{n}\cdot\nabla\phi+\phi\,\frac{1-P}{r}\,\mathbf{d}\cdot\mathbf{n}+P\,\mathbf{s}\cdot(\mathbf{n}\times\nabla\phi)+\phi\,\mathbf{s}\cdot(\mathbf{n}\times\nabla P)\right) \tag{17}$$

The potential representation (17) yields

$$\widetilde{\phi}=\frac{-1}{4\pi}\int_\Sigma \frac{d\mathcal{A}}{r}\,[\,\mathbf{n}\cdot\nabla\phi+\frac{\phi}{r}\,\mathbf{d}\cdot\mathbf{n}\,] \qquad \text{if } P=0 \tag{18a}$$

$$\widetilde{\phi}=\frac{-1}{4\pi}\int_\Sigma \frac{d\mathcal{A}}{r}\,[\,\mathbf{n}\cdot\nabla\phi+\mathbf{s}\cdot(\mathbf{n}\times\nabla\phi)\,] \qquad \text{if } P=1 \tag{18b}$$

The dipole term in the classical potential representation (18a) is $O(1/r^2)$. This term decays rapidly in the farfield but is strongly singular in the nearfield. The corresponding term in the alternative potential representation (18b) is $O(1/r)$. This term is weakly singular in the nearfield but decays slowly in the farfield. Thus, the alternative potential representations (18a) and (18b) are best suited in the farfield and the nearfield, respectively, and — in that sense — are complementary.

If the function P in (17) vanishes in the farfield and tends to 1 in the nearfield sufficiently rapidly, the integrand of (17) is asymptotically equivalent to the integrands of (18a) and (18b) in the farfield and the nearfield, respectively. E.g., consider the function

$$P=1/(1+r^3/\ell^3) \tag{19a}$$

where the positive real number ℓ corresponds to a transition length scale. This function yields

$$1-P=r^3/(\ell^3+r^3) \qquad \nabla P=3\,P\,r\,\mathbf{X}/(\ell^3+r^3) \tag{19b}$$

Expressions (17) and (19) yield

$$\widetilde{\phi}=\frac{-1}{4\pi}\int_\Sigma \frac{d\mathcal{A}}{r}\left[\mathbf{n}\cdot\nabla\phi+\frac{r^2\,\mathbf{d}\cdot\mathbf{n}}{\ell^3+r^3}\,\phi+\frac{\mathbf{s}\cdot(\mathbf{n}\times\nabla\phi)}{1+r^3/\ell^3}+\frac{3\,r^2\phi}{\ell^3+r^3}\,\frac{\mathbf{s}\cdot(\mathbf{d}\times\mathbf{n})}{1+r^3/\ell^3}\right] \tag{20}$$

where $\mathbf{s}$ and $\mathbf{d}$ are given by (16). The potential representation (20) is identical to the representations (18a) and (18b) in the limits $\ell=0$ and $\ell=\infty$, respectively. The integrand of (20) is identical to the integrands of (18a) and (18b) in the farfield and nearfield limits $r/\ell\to\infty$ and $r/\ell\to 0$, respectively.

6. Application to free-surface flows about ships or offshore structures

The boundary surface Σ and the Green function G (and related vector Green function $\mathbf{G}$) in the potential representation (12) are generic. This generic representation is now applied to free-surface flows, for which Σ is defined by (1). The unit vector $\mathbf{n}$ normal to Σ is given by $\mathbf{n}=(0\,,0\,,-1)$ and $\mathbf{n}=(0\,,0\,,1)$ at the free surface Σ_0 and the sea floor Σ_D , respectively. Thus, (12) and (10) yield

$$\widetilde{\phi}=\widetilde{\phi}_B+\widetilde{\phi}_0+\widetilde{\phi}_D \qquad \text{with} \tag{21a}$$

$$\widetilde{\phi}_B=\int_{\Sigma_B} d\mathcal{A}\,[\,G\,\mathbf{n}\cdot\nabla\phi-(1-P)\,\phi\,\mathbf{n}\cdot\nabla G+P\,\mathbf{G}\cdot(\mathbf{n}\times\nabla\phi)+\phi\,\mathbf{G}\cdot(\mathbf{n}\times\nabla P)\,] \tag{21b}$$

$$\widetilde{\phi}_0=\int_{\Sigma_0} dx\,dy\,[\,(P\phi)_x\,G^z_x+(P\phi)_y\,G^z_y+(1-P)\,\phi\,G_z-G\,\phi_z\,] \tag{21c}$$

$$\widetilde{\phi}_D=-\int_{\Sigma_D} dx\,dy\,[\,(P\phi)_x\,G^z_x+(P\phi)_y\,G^z_y+(1-P)\,\phi\,G_z\,] \tag{21d}$$

The boundary condition $\phi_z=0$ at the rigid sea floor Σ_D was used in (21d). The potential representation (21) is considered further on for diffraction-radiation by a ship advancing in time-harmonic waves.

7. The infinite-gravity and zero-gravity limits

Free-surface flows in the infinite-gravity and zero-gravity limits, associated with the boundary conditions $\phi_z=0$ (infinite gravity) and $\phi=0$ (zero gravity) at the plane $z=0$, are first considered. More generally, the nonhomogeneous problems corresponding to a specified vertical velocity ϕ_z or potential ϕ at the plane $z=0$ are considered. In the infinite-gravity limit, the Green function G is chosen to satisfy the boundary condition $G_z=0$ (and consequently also $G^z=0$ as verified further on) at $z=0$. The free-surface component (21c) therefore becomes

$$\widetilde{\phi}_0 = -\int_{\Sigma_0} dx\, dy\, G\,\phi_z \tag{22a}$$

This expression does not involve the function P. In the zero-gravity limit, the Green function G is chosen to satisfy the boundary condition $G = 0$ at $z = 0$, and the free-surface component (21c) becomes

$$\widetilde{\phi}_0 = \int_{\Sigma_0} dx\, dy\,[\,(1-P)\,\phi\, G_z + (P\phi)_x\, G^{\,z}_x + (P\phi)_y\, G^{\,z}_y\,] \tag{22b}$$

The Green function G may be chosen as $4\pi G^{\infty} = -1/r - 1/r_*$ for the infinite-gravity limit and as $4\pi G^{\,0} = -1/r + 1/r_*$ for the zero-gravity limit. Here, r is given by (13) and r_* is defined as

$$r_* = \sqrt{\mathbf{X}_* \cdot \mathbf{X}_*} \qquad \mathbf{X}_* = (\,X, Y, Z_*\,) \qquad Z_* = \widetilde{z} + z \tag{23}$$

X and Y are given by (2). The potentials $1/r$ and $1/r_*$ correspond to an elementary Rankine sink at a point $\mathbf{x} = (x\,, y\,, z)$ and at the mirror image $(x\,, y\,, -z)$ of $\mathbf{x}$ with respect to the free-surface plane $z = 0$, respectively. The function $(1/r_*)^z$ and its derivatives with respect to x and y are given by

$$(1/r_*)^z = \operatorname{sign}(Z_*)\, \ln(\, r_* + |\, Z_*\,|\,) \qquad \left\{ \begin{array}{c} (1/r_*)^z_x \\ (1/r_*)^z_y \end{array} \right\} = \frac{-\operatorname{sign}(Z_*)}{r_* + |\, Z_*\,|} \left\{ \begin{array}{c} X/r_* \\ Y/r_* \end{array} \right\} \tag{24}$$

with $\operatorname{sign}(Z_*) = -1$ and $|\, Z_*\,| = -Z_*$ in the lower half space $z \le 0$ and $\widetilde{z} \le 0$. At the plane $z = 0$, (15) and (24) yield $(1/r)^z = \ln(\, r - \widetilde{z}\,)$ and $(1/r_*)^z = -\ln(\, r - \widetilde{z}\,)$. Thus, the Green function G^{∞} satisfies the boundary condition $(G^{\infty})^z = 0$ at $z = 0$, as previously assumed.

In the infinite-gravity limit, (22a) with $G = G^{\infty}$ yields the free-surface component

$$\widetilde{\phi}_0 = \frac{1}{2\pi} \int_{\Sigma_0} \frac{dx\, dy}{r}\, \phi_z \tag{25a}$$

Similarly, (22b) with $G = G^{\,0}$, (15) and (24) yield

$$\widetilde{\phi}_0 = \frac{-1}{2\pi} \int_{\Sigma_0} \frac{dx\, dy}{r} \left(\frac{1-P}{r}\, \frac{\widetilde{z}\,\phi}{r} + \frac{d^{\,x}(P\phi)_x + d^{\,y}(P\phi)_y}{1 - \widetilde{z}/r} \right) \tag{25b}$$

in the zero-gravity limit. In (25), $r = \sqrt{X^2 + Y^2 + \widetilde{z}^{\,2}}$. The body-surface component (21b) becomes

$$\widetilde{\phi}_B = \frac{-1}{4\pi} \int_{\Sigma_B} d\mathcal{A} \left(\frac{a}{r} \pm \frac{a_*}{r_*} \right) \tag{26a}$$

where the upper/lower signs $+$ and $-$ in $\pm$ correspond to the infinite-gravity and zero-gravity limits, respectively, and the functions a and a_* are defined as

$$a = \mathbf{n} \cdot \nabla\phi + \phi\, \frac{1-P}{r}\, \mathbf{d} \cdot \mathbf{n} + P\, \mathbf{s} \cdot (\,\mathbf{n} \times \nabla\phi\,) + \phi\, \mathbf{s} \cdot (\,\mathbf{n} \times \nabla P\,) \tag{26b}$$

$$a_* = \mathbf{n} \cdot \nabla\phi + \phi\, \frac{1-P}{r_*}\, \mathbf{d}_* \cdot \mathbf{n} + P\, \mathbf{s}_* \cdot (\,\mathbf{n} \times \nabla\phi\,) + \phi\, \mathbf{s}_* \cdot (\,\mathbf{n} \times \nabla P\,) \tag{26c}$$

Furthermore, $\mathbf{s}$ and $\mathbf{d}$ are given by (16), and $\mathbf{s}_*$ and $\mathbf{d}_*$ are defined as

$$\mathbf{s}_* = \frac{\operatorname{sign}(d^{\,z}_*)}{1 + |d^{\,z}_*|}\, (d^{\,y}_*, -d^{\,x}_*, 0) \qquad (d^{\,x}_*, d^{\,y}_*, d^{\,z}_*) = \mathbf{d}_* = \frac{(x - \widetilde{x}, y - \widetilde{y}, z + \widetilde{z})}{r_*} \tag{27}$$

This definition of $\mathbf{d}_*$ yields $\nabla r_* = \mathbf{d}_*$ and $\nabla(1/r_*) = -\mathbf{d}_*/r_*^2$.

Expressions (25b) and (26) yield

$$\widetilde{\phi}_0 = \frac{-\widetilde{z}}{2\pi} \int_{\Sigma_0} \frac{dx\, dy}{r}\, \frac{\phi}{r^2} \tag{28a}$$

$$\widetilde{\phi}_B = \frac{-1}{4\pi} \int_{\Sigma_B} d\mathcal{A} \left[\left(\frac{1}{r} \pm \frac{1}{r_*} \right) \mathbf{n} \cdot \nabla\phi + \left(\frac{\mathbf{d} \cdot \mathbf{n}}{r^2} \pm \frac{\mathbf{d}_* \cdot \mathbf{n}}{r_*^2} \right) \phi \,\right] \tag{28b}$$

in the special case $P=0$, and

$$\widetilde{\phi}_0 = \frac{-1}{2\pi}\int_{\Sigma_0}\frac{dx\,dy}{r}\,\frac{d^{\,x}\phi_x + d^{\,y}\phi_y}{1-\widetilde{z}/r} \tag{29a}$$

$$\widetilde{\phi}_B = \frac{-1}{4\pi}\int_{\Sigma_B} d\mathcal{A}\left[\left(\frac{1}{r}\pm\frac{1}{r_*}\right)\mathbf{n}\cdot\nabla\phi + \left(\frac{\mathbf{s}}{r}\pm\frac{\mathbf{s}_*}{r_*}\right)\cdot(\mathbf{n}\times\nabla\phi)\right] \tag{29b}$$

in the special case $P=1$. The integrand of (28a) and the dipole term in (28b) decay rapidly as $r\to\infty$ but are strongly singular as $r\to 0$. The corresponding terms in (29a) and (29b) are weakly singular as $r\to 0$ but decay slowly as $r\to\infty$.

Substitution of (19) into (25b) and (26) yields

$$\widetilde{\phi}_0 = \frac{-1}{2\pi}\int_{\Sigma_0}\frac{dx\,dy}{r}\left[\frac{r\,\widetilde{z}\,\phi}{\ell^3+r^3} + \left(\frac{d^{\,x}\phi_x + d^{\,y}\phi_y}{1-\widetilde{z}/r} - 3\,r^2\phi\,\frac{1+\widetilde{z}/r}{\ell^3+r^3}\right)\frac{1}{1+r^3/\ell^3}\right] \tag{30a}$$

$$\widetilde{\phi}_B = \frac{-1}{4\pi}\int_{\Sigma_B} d\mathcal{A} \tag{30b}$$

$$\left[\left(\frac{1}{r}\pm\frac{1}{r_*}\right)\mathbf{n}\cdot\nabla\phi + \left(\frac{\mathbf{d}\cdot\mathbf{n}}{r^2}\pm\frac{\mathbf{d}_*\cdot\mathbf{n}}{r_*^2}\right)\frac{r^3\phi}{\ell^3+r^3} + \frac{\mathbf{s}/r\pm\mathbf{s}_*/r_*}{1+r^3/\ell^3}\cdot\left(\mathbf{n}\times\nabla\phi + \frac{3\,r^2\,\mathbf{d}\times\mathbf{n}}{\ell^3+r^3}\,\phi\right)\right]$$

Expressions (30) are identical to (28) and (29) in the limits $\ell=0$ and $\ell=\infty$, respectively. The integrands of (30a) and (30b) are identical to the corresponding integrands in (28) and (29) in the farfield and nearfield limits $r/\ell\to\infty$ and $r/\ell\to 0$, respectively. The contribution $\widetilde{\phi}_D$ of the sea floor Σ_D in (21) is easily obtained by taking the unit normal vector $\mathbf{n}$ in (30b) as $\mathbf{n}=(0,0,1)$. This sea-floor component can be rendered null if a more complicated Green function that satisfies the condition $G_z=0$ at the sea floor (in addition to the condition $G_z=0$ or $G=0$ at the free surface) is used.

8. Free-surface contribution for wave diffraction-radiation with forward speed

Diffraction-radiation by a ship advancing (at speed $\mathcal{U}$) through regular waves (with frequency ω) is now considered. Define the nondimensional wave frequency f, the Froude number F, and $\widehat{\tau}$ as

$$f=\omega\sqrt{L/g} \qquad F=\mathcal{U}/\sqrt{gL} \qquad \widehat{\tau}=2\,fF=2\,\omega\,\mathcal{U}/g \tag{31}$$

Furthermore, define π^ϕ and π^G as

$$\pi^\phi = \phi_z + F^2\phi_{xx} - f^2\phi + i\widehat{\tau}\,\phi_x \tag{32a}$$

$$\pi^G = G_z + F^2 G_{xx} - f^2 G - i\widehat{\tau}\,G_x \tag{32b}$$

The integrand of the free-surface integral (21c) can be expressed as

$$(P\phi)_x\,(\pi^G)^{zz}_x + (P\phi)_y\,(\pi^G)^{zz}_y + (1-P)\,\phi\,\pi^G - G\,\pi^\phi + f^2 a^f + i\widehat{\tau}\,a^\tau + F^2 a^F$$

where π^G and π^ϕ are given by (32) and a^f, a^τ, a^F are defined as

$$a^f = (P\phi\,G^{zz}_x)_x + (P\phi\,G^{zz}_y)_y$$
$$a^\tau = [(P\phi)_y\,G^{zz}_y]_x - [(P\phi)_x\,G^{zz}_y]_y + [(1-P)\,\phi\,G]_x$$
$$a^F = (\phi_x\,G)_x - [(P\phi)_y\,G^{zz}_{xy}]_x + [(P\phi)_x\,G^{zz}_{xy}]_y - [(1-P)\,\phi\,G_x]_x$$

Stokes' theorem can then be used to express the free-surface integral (21c) as

$$\begin{aligned}\widetilde{\phi}_0 = &\int_{\Sigma_0} dx\,dy\,[(\pi^G)^{zz}_x(P\phi)_x + (\pi^G)^{zz}_y(P\phi)_y + \pi^G(1-P)\,\phi - G\,\pi^\phi\,]\\ &+\int_\Gamma d\mathcal{L}\,[f^2(t^yG^{zz}_x - t^xG^{zz}_y)\,P\phi - (F^2G_x - i\widehat{\tau}G)^{zz}_y\,\mathbf{t}\cdot\nabla(P\phi)\\ &\qquad - (F^2G_x - i\widehat{\tau}G)(1-P)\,t^y\phi + F^2\,G\,t^y\phi_x\,]\end{aligned} \tag{33}$$

In the line integral around the curve Γ, $\mathbf{t}\cdot\nabla\phi$ is the velocity along the unit vector $\mathbf{t}=(t^x,t^y,0)$ tangent to Γ (oriented clockwise; looking down), and the velocity component ϕ_x can be expressed as

$$\phi_x = t^x\,\mathbf{t}\cdot\nabla\phi - t^y\,\mathbf{n}^\Gamma\cdot\nabla\phi \tag{34}$$

with $\mathbf{n}^\Gamma=(-t^y,t^x,0)$ a unit vector normal to the curve Γ in the free-surface plane $z=0$.

9. Potential representation for wave diffraction-radiation with forward speed

Substitution of (33) into (21) yields the potential representation

$$\widetilde{\phi} = \int_{\Sigma_B} d\mathcal{A}\,(G\,\mathbf{n}\cdot\nabla\phi + A_B) + \int_{\Sigma_0} dx\,dy\,(A_0 - G\,\pi^{\phi}) + \int_{\Gamma} d\mathcal{L}\,(A_\Gamma + F^2\,G\,t^y\phi_x) - \int_{\Sigma_D} dx\,dy\,A_D \tag{35}$$

where the amplitude functions A_B , A_0 , A_Γ and A_D are defined as

$$A_B = \mathbf{G}\cdot[\,\mathbf{n}\times\nabla(P\phi)] - (1-P)\,\phi\,\mathbf{n}\cdot\nabla G \tag{36a}$$

$$A_0 = (\pi^G)^{zz}_x\,(P\phi)_x + (\pi^G)^{zz}_y\,(P\phi)_y + \pi^G(1-P)\,\phi \tag{36b}$$

$$A_\Gamma = f^2\,(t^y G^{zz}_x - t^x G^{zz}_y)P\phi - (F^2 G_x - i\,\widehat{\tau}\,G)^{zz}_y\,\mathbf{t}\cdot\nabla(P\phi)$$
$$\qquad - (F^2 G_x - i\,\widehat{\tau}\,G)(1-P)\,t^y\phi \tag{36c}$$

$$A_D = G^z_x\,(P\phi)_x + G^z_y\,(P\phi)_y + G_z\,(1-P)\,\phi \tag{36d}$$

The function π^{ϕ} defined by (32a) is null if the potential ϕ is assumed to satisfy the usual linearized free-surface boundary condition. However, π^{ϕ} is not null if a pressure distribution is applied at the free surface, as for a surface-effect ship, or if nearfield free-surface effects are taken into account, e.g. for linearization about a base flow (double body or steady flow) that differs from the uniform stream opposing the ship speed (for which $\pi^{\phi} = 0$). In any case, the pressure/flux distribution π^{ϕ} at the free surface Σ_0 in (35) vanishes in the farfield. If the Green function G in the potential representation (35) is chosen as the usual Green function associated with the linearized free-surface boundary condition, the function π^G given by (32b) is null. Alternatively, if a Green function that satisfies the free-surface condition $\pi^G = 0$ in the farfield (but not in the nearfield) is used, the function π^G and the related amplitude function A_0 given by (36b) vanish in the farfield, like the function π^{ϕ}. In either case, free-surface integration in (35) is only required over a finite nearfield region of the unbounded free surface Σ_0 .

10. Two Green functions and related potential representations

Two alternative Green functions are defined in *Noblesse and Yang (2004b)* and considered here. These Green functions can be expressed as

$$4\pi G = G^R + G^F \qquad 4\pi G = G^R + i\,G^W \tag{37}$$

The Green function $G^R + G^F$ satisfies the linear free-surface boundary condition $\pi^G = 0$ everywhere, i.e. in both the farfield — where the linear free-surface condition $\pi^G = 0$ is valid — and the nearfield, where this linear condition is only an approximation. The Green function $G^R + i\,G^W$ satisfies the linear free-surface condition $\pi^G = 0$ accurately in the farfield but only approximately in the nearfield. The component G^R in (37) stands for a local-flow component given by elementary free-space Rankine sources. The component G^F represents a Fourier component defined by a two-dimensional Fourier superposition of elementary waves, given further on in this study for deep water. Finally, the component G^W in (37) stands for a wave component defined by $G^W = W^+ - W^- - W^i$, where the three components $W^{\pm}$ and W^i represent distinct wave systems generated by a pulsating source advancing at constant speed. Each of these three wave components is defined by a one-dimensional Fourier superposition of elementary waves, given in *Noblesse and Yang (2004b)* for deep water.

Substitution of the alternative decompositions (37) of the Green function into the potential representation (35) yields the alternative representations

$$4\pi\,\widetilde{\phi} = \widetilde{\phi}^R + \widetilde{\phi}^F \qquad 4\pi\,\widetilde{\phi} = \widetilde{\phi}^R + i\,\widetilde{\phi}^W \tag{38}$$

for the potential $\widetilde{\phi}$ at a field point $\widetilde{\mathbf{x}}$ within the flow domain. The potentials $\widetilde{\phi}^R, \widetilde{\phi}^F$ and $\widetilde{\phi}^W$ in (38) are defined by the basic potential representation (3), with G taken as G^R, G^F and G^W, respectively. These basic potential representations can be modified using the transformation (11b), as in (12), (21) and (35). Each one of the three components G^R, G^F and G^W in the alternative Green functions (37) satisfies the Laplace equation. The transformation (11b) can therefore be applied separately to the local Rankine component $\widetilde{\phi}^R$, the Fourier component $\widetilde{\phi}^F$ and the wave component $\widetilde{\phi}^W$ in the alternative decompositions (38), and these three components of the potential $\widetilde{\phi}$ are given by (35), with G replaced by G^R, G^F or G^W. These three potentials are successively considered below.

11. Rankine potential $\widetilde{\phi}^R$

Thus, the Rankine component $\widetilde{\phi}^R$ in (38) is defined by (35) and (36) as

$$\widetilde{\phi}^R = \int_{\Sigma_B} d\mathcal{A}\,(G^R \mathbf{n}\cdot\nabla\phi + A_B^R) + \int_{\Sigma_0} dx\,dy\,(A_0^R - G^R\pi^\phi) + \int_\Gamma d\mathcal{L}\,(A_\Gamma^R + F^2 G^R t^y \phi_x) - \int_{\Sigma_D} dx\,dy\,A_D^R \tag{39}$$

where

$$\begin{aligned}
A_B^R &= \mathbf{G}^R\cdot[\,\mathbf{n}\times\nabla(P\phi)] - (1-P)\,\phi\,\mathbf{n}\cdot\nabla G^R \\
A_D^R &= (G^R)_x^z\,(P\phi)_x + (G^R)_y^z\,(P\phi)_y + G_z^R(1-P)\,\phi \\
A_0^R &= (\pi^R)_x^{zz}\,(P\phi)_x + (\pi^R)_y^{zz}\,(P\phi)_y + \pi^R(1-P)\,\phi \\
A_\Gamma^R &= f^2\,[\,t^y(G^R)_x^{zz} - t^x(G^R)_y^{zz}\,]P\phi - (F^2G_x^R - i\widehat{\tau}G^R)_y^{zz}\,\mathbf{t}\cdot\nabla(P\phi) \\
&\quad -(F^2G_x^R - i\widehat{\tau}G^R)(1-P)\,t^y\phi
\end{aligned} \tag{40}$$

with $\mathbf{G}^R = [\,(G^R)_y^z\,, -(G^R)_x^z\,, 0\,]$ and $\pi^R = G_z^R + F^2 G_{xx}^R - f^2 G^R - i\widehat{\tau}\,G_x^R$ (41)

in accordance with (10) and (32b). Expressions (40) yield

$$A_B^R = -\phi\,\mathbf{n}\cdot\nabla G^R \qquad A_D^R = G_z^R\,\phi \qquad A_0^R = \pi^R\phi \qquad A_\Gamma^R = -(F^2G_x^R - i\widehat{\tau}G^R)\,t^y\phi \tag{42a}$$

in the special case $P = 0$, and

$$\begin{aligned}
&A_B^R = \mathbf{G}^R\cdot(\mathbf{n}\times\nabla\phi) \qquad A_D^R = (G^R)_x^z\,\phi_x + (G^R)_y^z\,\phi_y \qquad A_0^R = (\pi^R)_x^{zz}\,\phi_x + (\pi^R)_y^{zz}\,\phi_y \\
&A_\Gamma^R = f^2\,[\,t^y(G^R)_x^{zz} - t^x(G^R)_y^{zz}\,]\,\phi - (F^2G_x^R - i\widehat{\tau}G^R)_y^{zz}\,\mathbf{t}\cdot\nabla\phi
\end{aligned} \tag{42b}$$

in the special case $P = 1$. Expressions (42a) and (42b) correspond to the classical potential representation (3) and the weakly-singular representation (8), respectively. The functions $A_B^R\,, A_D^R\,, A_0^R\,, A_\Gamma^R$ defined by (42a) vanish faster in the farfield — but are more singular in the nearfield — than the functions (42b). Thus, the functions (42a) and (42b) are better suited in the farfield and the nearfield, respectively.

Substitution of (19) into (40) yields

$$\begin{aligned}
A_B^R &= \frac{\mathbf{G}^R}{1+r^3/\ell^3}\cdot\left(\mathbf{n}\times\nabla\phi + \frac{3\,r\,\mathbf{n}\times\mathbf{X}}{\ell^3+r^3}\,\phi\right) - \frac{r^3\phi\,\mathbf{n}\cdot\nabla G^R}{\ell^3+r^3} \\
A_D^R &= \frac{r^3G_z^R\phi}{\ell^3+r^3} + \frac{(G^R)_x^z}{1+r^3/\ell^3}\left(\phi_x + \frac{3\,r\,X\phi}{\ell^3+r^3}\right) + \frac{(G^R)_y^z}{1+r^3/\ell^3}\left(\phi_y + \frac{3\,r\,Y\phi}{\ell^3+r^3}\right) \\
A_0^R &= \frac{r^3\pi^R\phi}{\ell^3+r^3} + \frac{(\pi^R)_x^{zz}}{1+r^3/\ell^3}\left(\phi_x + \frac{3\,r\,X\phi}{\ell^3+r^3}\right) + \frac{(\pi^R)_y^{zz}}{1+r^3/\ell^3}\left(\phi_y + \frac{3\,r\,Y\phi}{\ell^3+r^3}\right) \\
A_\Gamma^R &= f^2\phi\,\frac{t^y(G^R)_x^{zz} - t^x(G^R)_y^{zz}}{1+r^3/\ell^3} - (F^2G_x^R - i\widehat{\tau}G^R)\frac{r^3t^y\phi}{\ell^3+r^3} \\
&\quad - \frac{(F^2G_x^R - i\widehat{\tau}G^R)_y^{zz}}{1+r^3/\ell^3}\left(\mathbf{t}\cdot\nabla\phi + \frac{3\,r\,\mathbf{t}\cdot\mathbf{X}}{\ell^3+r^3}\,\phi\right)
\end{aligned} \tag{43}$$

Expressions (43) are identical to (42a) and (42b) in the limits $\ell = 0$ and $\ell = \infty$, and are asymptotically equivalent to (42a) and (42b) in the farfield $r/\ell \to \infty$ and the nearfield $r/\ell \to 0$, respectively, and thus are well suited in both the farfield and the nearfield.

12. Fourier potential $\widetilde{\phi}^F$

As already noted, the component G^F in (37) stands for a Fourier component defined by a two-dimensional Fourier superposition of elementary waves. Specifically, in the deep-water limit — now considered — the Fourier component G^F is given by

$$G^F = \lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{-\infty}^{\infty}\!d\beta\int_{-\infty}^{\infty}\!d\alpha\,A\,\frac{e^{Z_*\,k - i\,(\,\alpha X + \beta Y\,)}}{D + i\,\varepsilon\,D_1} \tag{44a}$$

where X, Y and Z_* are given by (2) and (23), and the dispersion functions D and D_1 and the amplitude function A are defined as

$$D = k - (F\alpha - f)^2 \qquad D_1 = F\alpha - f \qquad A = e^{-F^2k}(1 - e^{-k/f^2})\,D/k - 1 \tag{44b}$$

The Fourier component $\widetilde{\phi}^F$ in (38) is given by (35) and (36) with G taken as G^F. The Fourier-Kochin approach can be used to express the Fourier potential $\widetilde{\phi}^F$ as

$$\widetilde{\phi}^F = \lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \, A\, S\, \frac{e^{\widetilde{z}\,k - i\,(\widetilde{x}\,\alpha + \widetilde{y}\,\beta)}}{D + i\,\varepsilon\, D_1} \tag{45a}$$

Here, (44a), (2) and (23) were used, the dispersion functions D and D_1 and the amplitude function A are given by (44b), and $S(\alpha\,,\beta)$ stands for the spectrum function defined as

$$S = \int_{\Sigma_B} d\mathcal{A}(\mathbf{n}\cdot\nabla\phi + i\,A_B^F)\,e^{k\,z}E - \int_{\Sigma_0} dx\,dy\,(\pi^\phi - i\,A_0^F\,\frac{D}{k})\,E + \int_\Gamma d\mathcal{L}\,(F^2 t^y \phi_x + i\,A_\Gamma^F)\,E \tag{45b}$$

with

$$E = e^{\,i\,(\,\alpha\,x + \beta\,y\,)} \tag{45c}$$

Expressions (35), (36), (10), (32b) and (44b) show that the amplitude functions $A_B^F\,, A_0^F, A_\Gamma^F$ in (45b) are given by

$$\begin{aligned}
A_B^F &= \frac{\beta}{k}\,[\mathbf{n}\times\nabla(P\phi)]^x - \frac{\alpha}{k}\,[\mathbf{n}\times\nabla(P\phi)]^y - (\alpha\,n^x + \beta\,n^y - i\,k\,n^z)(1-P)\,\phi \\
A_0^F &= \frac{\alpha}{k}\,(P\phi)_x + \frac{\beta}{k}\,(P\phi)_y - i\,k\,(1-P)\,\phi \\
A_\Gamma^F &= \frac{f^2 P\phi}{k}\left(\frac{\alpha}{k}\,t^y - \frac{\beta}{k}\,t^x\right) + \frac{\widehat{\tau} - F^2\alpha}{k}\left(k\,(1-P)\,t^y\phi + i\,\frac{\beta}{k}\,\mathbf{t}\cdot\nabla(P\phi)\right)
\end{aligned} \tag{46}$$

The amplitude functions (46) become

$$A_B^F = -(\alpha n^x + \beta n^y - i\,k\,n^z)\,\phi \qquad A_0^F = -i\,k\,\phi \qquad A_\Gamma^F = (\widehat{\tau} - F^2\alpha)\,t^y\phi \tag{47a}$$

in the special case $P = 0$, and

$$\begin{aligned}
A_B^F &= \frac{\beta}{k}\,(\mathbf{n}\times\nabla\phi)^x - \frac{\alpha}{k}\,(\mathbf{n}\times\nabla\phi)^y \qquad A_0^F = \frac{\alpha}{k}\,\phi_x + \frac{\beta}{k}\,\phi_y \\
A_\Gamma^F &= \frac{f^2\phi}{k}\left(\frac{\alpha}{k}\,t^y - \frac{\beta}{k}\,t^x\right) + i\,\frac{\beta}{k}\,\frac{\widehat{\tau} - F^2\alpha}{k}\,\mathbf{t}\cdot\nabla\phi
\end{aligned} \tag{47b}$$

in the special case $P = 1$. Expressions (47a) and (47b) correspond to the classical potential representation (3) and the weakly-singular representation (8), respectively. Expressions (47) yield

$$\left\{\begin{array}{lll} A_B^F = O(k) & A_0^F = O(k) & \text{if } P = 0 \\ A_B^F = O(1) & A_0^F = O(1) & \text{if } P = 1 \end{array}\right\}$$

in both the limit $k \to 0$ and the limit $k \to \infty$. Expressions (47) also yield

$$\begin{array}{llll}
A_\Gamma^F = \left\{\begin{array}{ll} O(1) & \text{if } P = 0 \\ O(1/k) & \text{if } P = 1 \end{array}\right\} \text{if } f \neq 0 & A_\Gamma^F = \left\{\begin{array}{ll} O(k) & \text{if } P = 0 \\ O(1) & \text{if } P = 1 \end{array}\right\} \text{if } f = 0 & \text{as } k \to 0 \\
A_\Gamma^F = \left\{\begin{array}{ll} O(k) & \text{if } P = 0 \\ O(1) & \text{if } P = 1 \end{array}\right\} \text{if } F \neq 0 & A_\Gamma^F = \left\{\begin{array}{ll} O(1) & \text{if } P = 0 \\ O(1/k) & \text{if } P = 1 \end{array}\right\} \text{if } F = 0 & \text{as } k \to \infty
\end{array}$$

These asymptotic approximations show that the amplitude functions $A_B^F\,, A_0^F$ and A_Γ^F for $P = 0$ are smaller than the corresponding amplitude functions for $P = 1$ in the limit $k \to 0$, and that the reverse holds in the limit $k \to \infty$. Thus, the functions (47a) and (47b), which correspond to the classical and weakly-singular potential representations, are preferable in the limits $k \to 0$ and $k \to \infty$, respectively. This property agrees with the previously-established property that the classical and weakly-singular potential representations are better suited in the farfield and the nearfield, respectively, since the farfield and nearfield behavior of a function is determined by the behavior of its Fourier transform in the limits $k \to 0$ and $k \to \infty$, respectively.

Substitution of the weight function

$$P = k^2/(k^2 + k_*^2) \tag{48}$$

where the positive real number k_* stands for a transition wavenumber, into (46) yields

$$(A_B^F\,,A_0^F\,,A_\Gamma^F\,) = \frac{k\,k_*}{k^2\!+k_*^2}\,(a_B^F\,,a_0^F\,,a_\Gamma^F\,) \qquad \text{with} \tag{49a}$$

$$a_B^F = \frac{\beta}{k_*}(\mathbf{n}\times\nabla\phi)^x - \frac{\alpha}{k_*}(\mathbf{n}\times\nabla\phi)^y - \left(\frac{\alpha}{k}\,n^x + \frac{\beta}{k}\,n^y - i\,n^z\right)k_*\,\phi \tag{49b}$$

$$a_0^F = \frac{\alpha}{k_*}\,\phi_x + \frac{\beta}{k_*}\,\phi_y - i\,k_*\,\phi \tag{49c}$$

$$a_\Gamma^F = \frac{f^2\phi}{k_*}\left(\frac{\alpha}{k}\,t^y - \frac{\beta}{k}\,t^x\right) + \frac{\widehat{\tau}-F^2\alpha}{k}\left(k_*\,t^y\phi + i\,\frac{\beta}{k_*}\,\mathbf{t}\cdot\nabla\phi\right) \tag{49d}$$

Expressions (49) are identical to (47a) and (47b) in the limits $k_* = \infty$ and $k_* = 0$, and are asymptotically equivalent to (47a) and (47b) in the limits $k/k_* \to 0$ and $k/k_* \to \infty$, respectively.

13. Wave potential $\widetilde{\phi}^W$

The Fourier-Kochin representation (45a) is a singular double Fourier integral, a major difficulty. This basic difficulty is circumvented in the Fourier-Kochin representation of the wave component $\widetilde{\phi}^W$ in (38), which is given by three nonsingular single (one-fold) Fourier integrals as shown below. As already noted, the wave potential $\widetilde{\phi}^W$ is related to the Green function

$$4\pi\,G = G^R + i\,G^W = G^R + i\,(W^+ - W^- - W^i) \tag{50a}$$

where G^R represents a local-flow component given by four elementary Rankine sources, and the three components $W^\pm$ and W^i represent distinct wave systems generated by a pulsating source advancing at constant speed. Substitution of the Green function (50a) into the potential representation (35) yields

$$4\pi\,\widetilde{\phi} = \widetilde{\phi}^R + i\,\widetilde{\phi}^W = \widetilde{\phi}^R + i\,(\widetilde{\phi}^{\,+} - \widetilde{\phi}^{\,-} - \widetilde{\phi}^{\,i}\,) \tag{50b}$$

For deep water and in the farfield — now considered — the wave components $W^\pm$ and W^i in (50a) are given by single Fourier integrals of the form

$$W = \Lambda\!\int_{-T}^{T}\!dt\;\Omega\,\Theta_0\,e^{\,Z_*\,k\,-\,i\,(X\,\alpha\,+Y\beta)} \quad \text{with} \quad \Theta_0 = 1 + \epsilon\,\text{sign}(X\alpha + Y\beta) \tag{51}$$

X, Y and Z_* are defined by (2) and (23). Furthermore, $\Lambda = \Lambda(f, F)$ is a function of f and F, the limit of integration T is equal to $\infty\,,\pi\,,$ or a function of τ (thus, T is independent of X, Y, Z_*), the amplitude function $\Omega = \Omega(t\,;\tau)$ is a function of t and τ, $\epsilon = \pm 1$, and the Fourier variables α and β and the related wavenumber k are functions of f, F and t. The factor Λ, the limit of integration T, the amplitude function Ω, the value of ϵ, and the functions $\alpha\,,\beta$ and k are given in *Noblesse and Yang (2004b)* for each of the three wave components $W^\pm$ and W^i in (50a). In the nearfield, the sign function $\text{sign}(X\alpha + Y\beta)$ in (50a) is replaced by a function Θ, given in *Noblesse and Yang (2004b)*.

The wave potentials $\widetilde{\phi}^{\,\pm}$ and $\widetilde{\phi}^{\,i}$ in (50b) are given by (35) and (36), with G replaced by $W^\pm$ and W^i, respectively. The Fourier-Kochin approach, already used for the Fourier component $\widetilde{\phi}^F$, can be used again to express the wave potentials $\widetilde{\phi}^{\,\pm}$ and $\widetilde{\phi}^{\,i}$ in (50b) as single Fourier integrals (along the dispersion curves defined by the dispersion relation $D = 0$) that involve the spectrum function (45b) at the dispersion curves $D = 0$. Thus, in the farfield, the wave potentials $\widetilde{\phi}^{\,\pm}$ and $\widetilde{\phi}^{\,i}$ are given by

$$\phi_\infty^{+/-/i} = \Lambda\!\int_{-T}^{T}\!dt\;\Omega\,S_\infty^W\,e^{\,\widetilde{z}\,k\,-\,i\,(\,\widetilde{x}\,\alpha\,+\,\widetilde{y}\,\beta\,)} \tag{52a}$$

where S_∞^W stands for the farfield wave spectrum function

$$S_\infty^W = \int_{\Sigma_B}\!d\mathcal{A}\,\Theta_0\,(\mathbf{n}\cdot\nabla\phi + i\,A_B^F)\,e^{\,k\,z}E - \int_{\Sigma_0}\!dx\,dy\;\Theta_0\,\pi^\phi E + \int_\Gamma\!d\mathcal{L}\;\Theta_0\,(F^2\,t^y\phi_x + i\,A_\Gamma^F\,)\,E \tag{52b}$$

with E and π^ϕ defined by (45c) and (32a), respectively. If expression (48) for the weight function P is used, the functions A_B^F and A_Γ^F are given by (49a), (49b) and (49d). These expressions yield

$$A_B^F = -(\alpha\,n^x + \beta\,n^y - i\,k\,n^z\,)\,\phi \qquad\qquad A_\Gamma^F = (\widehat{\tau} - F^2\alpha)\,t^y\phi \tag{52c}$$

$$A_B^F = \frac{\beta}{k}(\mathbf{n}\times\nabla\phi)^x - \frac{\alpha}{k}(\mathbf{n}\times\nabla\phi)^y \qquad\qquad A_\Gamma^F = \frac{f^2\phi}{k}\left(\frac{\alpha}{k}\,t^y - \frac{\beta}{k}\,t^x\right) + i\,\frac{\beta}{k}\,\frac{\widehat{\tau}-F^2\alpha}{k}\,\mathbf{t}\cdot\nabla\phi \tag{52d}$$

in the special cases $k_* = \infty$ and $k_* = 0$, in agreement with (47).

14. Optimal wave spectrum functions

The farfield wave spectrum function (52b) can be expressed as

$$S_\infty^W = S_\psi^W + S_B^W + S_\Gamma^W \qquad \text{with} \tag{53a}$$

$$S_\psi^W = \int_{\Sigma_B} d\mathcal{A}\, \Theta_0 \,(\mathbf{n}\cdot\nabla\phi)\, e^{k\,z} E - \int_{\Sigma_0} dx\, dy\, \Theta_0\, \pi^\phi E \tag{53b}$$

$$S_B^W = i \int_{\Sigma_B} d\mathcal{A}\, \Theta_0\, A_B^F\, e^{k\,z} E \qquad S_\Gamma^W = \int_\Gamma d\mathcal{L}\, \Theta_0\,(F^2 t^y \phi_x + i\, A_\Gamma^F) E \tag{53c}$$

The component S_ψ^W, associated with the normal flux $\mathbf{n}\cdot\nabla\phi$ at the body surface Σ_B and the pressure/flux distribution π^ϕ at the free surface Σ_0 in (52b), does not involve the transition wavenumber k_*. The components S_Γ^W and S_B^W are functions of k_*, and represent the contributions of the line integral along Γ and of the term A_B^F in the surface integral over Σ_B, respectively. Specifically, (49a), (49b) and (49d) show that the amplitude functions in the integrands of the integrals defined by (53c) are of the form

$$\frac{k\,k_*}{k^2+k_*^2}\left(\frac{P}{k_*} + Q\,k_*\right) + R = k\,\frac{P-k^2 Q}{k^2+k_*^2} + k\,Q + R$$

where P, Q and R do not involve k_*. The derivative of the foregoing function with respect to k_* is

$$-2\,k\,(P-k^2 Q)\,k_*/(k^2+k_*^2)^2$$

The derivatives of the functions S_B^W and S_Γ^W with respect to k_* therefore vanish for both $k_* = 0$ and $k_* = \infty$. Thus, the components S_B^W and S_Γ^W are largest or smallest for $k_* = 0$ or $k_* = \infty$, and numerical cancellations between these two components likewise are largest or smallest if $k_* = 0$ or $k_* = \infty$. In this sense, the optimal representation of the farfield wave spectrum function S_∞^W in (52a) is given by (52b) with either (52c) or (52d). These two alternative expressions correspond to $k_* = \infty$ or $k_* = 0$, i.e. to the classical potential representation (3) or the weakly-singular representation (8), respectively.

In the particular case $F = 0$, i.e. for wave diffraction-radiation without forward speed, (52c) yields $A_\Gamma^F = 0$, and the line integral S_Γ^W defined by (53c) is null. Thus, no numerical cancellation can occur between the components S_Γ^W and S_B^W in (53a) if $k_* = \infty$; and the classical representation (52c) is in this sense preferable in the particular case $F = 0$. In the particular case $f = 0$, i.e. for steady flow, the dispersion curve $D = 0$, defined by (44b) as $F^2\alpha^2 = k$, yields $\alpha = \sqrt{k}/F$ and $\beta \sim k$ as $k \to \infty$. In the limit $k \to \infty$, the alternative amplitude functions A_B^F defined by (52c) and (52d) therefore are $O(k)$ and $O(1)$, respectively, and the related alternative amplitude functions $F^2 t^y \phi_x + i\, A_\Gamma^F$ are $O(\sqrt{k}\,)$ and $O(1)$, respectively. The components S_B^W and S_Γ^W in (53a) can then be expected to be significantly larger for (52c) than for (52d) as $k \to \infty$, as can in fact be observed in the numerical example considered in *Noblesse and Yang (2004a)*. Thus, significantly larger numerical cancellations between the contributions of the body surface Σ_B and the curve Γ can be expected for the classical representation (52c) than for the weakly-singular representation (52d), which thus is preferable for the particular case $f = 0$.

For the general case $Ff \neq 0$, i.e. for wave diffraction-radiation with forward speed, the wave potentials $\widetilde{\phi}^{\,i}$ and $\widetilde{\phi}^{\,\pm}$ in (50b) are associated with the inner and outer dispersion curves I and $O^\pm$, respectively, defined in *Noblesse and Yang (2004b)*. The wavenumber k and the related Fourier variables α and β are bounded for the inner dispersion curve I but are unbounded for the outer dispersion curves $O^\pm$. Specifically, we have $k \le f/F$ and $f/F \le k$ for the inner dispersion curve I and the outer dispersion curves $O^\pm$, respectively. Thus, the classical representation (52c) and the weakly-singular representation (52d) can be expected to be preferable for the wave potentials $\widetilde{\phi}^{\,i}$ and $\widetilde{\phi}^{\,\pm}$, respectively, in (50b).

15. Conclusion

The classical potential representation (3) and the weakly-singular representation (8) are best suited in the farfield and the nearfield, respectively. This property is clearly apparent if the Green function G in (3) and (8) is chosen as the fundamental free-space Rankine source given by $4\pi G = -1/r$, for which the alternative potential representations (3) and (8) become (18a) and (18b). Specifically, the

$O(1/r^2)$ dipole term in the classical potential representation (18a) decays rapidly in the farfield but is strongly singular in the nearfield, whereas the corresponding $O(1/r)$ term in the alternative potential representation (18b) is weakly singular in the nearfield but decays slowly in the farfield. Thus, the classical potential representation (3) and the weakly-singular representation (8) are complementary.

These alternative basic representations of the potential can be regarded as special cases of the generalized representation (12). Specifically, the representations (3) and (8) are obtained if the weight function P in the generalized representation (12) is chosen as $P = 0$ or $P = 1$, respectively. If the Green function G is taken as $4\pi G = -1/r$ and the weight function P given by (19) is chosen, the generalized representation (12) becomes (20). The integrand of the latter representation is identical to the integrands of (18a) and (18b) in the farfield and nearfield limits $r/\ell \to \infty$ and $r/\ell \to 0$, respectively. Thus, the generalized representations (20) and (12) are weakly singular — like the weakly-singular representation (8) — and define a potential $\widetilde{\phi}$ at a flow-field point $\widetilde{\mathbf{x}}$ that is continuous at the boundary surface Σ, whereas the potential $\widetilde{\phi}$ defined by the classical representation (3) is not continuous at Σ.

The generalized potential representation (12) has been applied to free-surface flows in the infinite-gravity and zero-gravity limits, and to wave diffraction-radiation by a ship advancing through regular waves in uniform finite water depth. In the latter case, the generalized potential representation (12) becomes (35) with (36). This representation does not presume that the potential ϕ satisfies the linearized free-surface boundary condition $\pi^\phi = 0$. Indeed, the free-surface flux π^ϕ in (35), given by (32a), can account for nearfield effects associated with linearization about a base flow (e.g. double-body flow or steady free-surface flow) that differs from the uniform stream opposing the ship speed (for which $\pi^\phi = 0$).

The potential representation (35) defines the potential $\widetilde{\phi}$ for wave diffraction-radiation with forward speed in terms of boundary distributions over the body (ship hull) surface Σ_B, the sea floor Σ_D, the free surface Σ_0, and the intersection curve Γ between Σ_B and Σ_0. The distribution over the body surface Σ_B involves the potential ϕ and the velocity components $\mathbf{n}\cdot\nabla\phi$ and $\mathbf{n}\times\nabla\phi$, which are normal and tangent to Σ_B. The distribution over the sea floor Σ_D involves the potential ϕ and the tangential velocity components ϕ_x and ϕ_y. The distribution over the free surface Σ_0 involves the pressure/flux function π^ϕ given by (32a), the potential ϕ, and the tangential velocity components ϕ_x and ϕ_y. Finally, the distribution around the curve Γ involves the potential ϕ and the velocity components $\mathbf{t}\cdot\nabla\phi$ and ϕ_x, where $\mathbf{t}\cdot\nabla\phi$ is the velocity component along the unit vector $\mathbf{t}$ tangent to Γ, and the velocity component ϕ_x can be expressed in the form given by (34).

Two alternative Green functions related to the linearized free-surface boundary condition $\pi^G = 0$ for wave diffraction-radiation with forward speed have been used. One of these two Green functions is the usual free-surface Green function, which satisfies $\pi^G = 0$ in both the farfield (where the linear free-surface condition $\pi^G = 0$ is valid) and the nearfield, where this linear condition is only an approximation. The other Green function, given in *Noblesse and Yang (2004b)*, satisfies the linear free-surface condition $\pi^G = 0$ accurately in the farfield but only approximately in the nearfield. Use of these two alternative Green functions, expressed as in (37), in the potential representation (35) yields the corresponding alternative potential representations (38). These potential representations express the potential $\widetilde{\phi}$ as the sum of a local-flow Rankine component $\widetilde{\phi}^R$ and a Fourier component $\widetilde{\phi}^F$ or a wave component $\widetilde{\phi}^W$.

Use of the simple Green function given in *Noblesse and Yang (2004b)* leads to the free-surface function A_0 in the potential representation (35). The function A_0, defined by (36b), is related to the property that the function π^G given by (32b) is not null in the nearfield for the simple Green function, as already noted (whereas π^G and A_0 are null if the usual free-surface Green function is used). Thus, use of the simple Green function given in *Noblesse and Yang (2004b)*, instead of the usual free-surface Green function, implies no approximation or restriction, but requires the distribution A_0 over a nearfield portion of the free surface Σ_0.

The Rankine component G^R in the alternative Green functions (37) is defined in *Noblesse and Yang (2004b)* by four elementary free-space Rankine sources. Accordingly, the Rankine potential $\widetilde{\phi}^R$ in (38) is defined by (39) and (43) in terms of distributions of elementary Rankine singularities over the boundary $\Sigma_B \cup \Gamma \cup \Sigma_0 \cup \Sigma_D$. Expressions (43) involve several functions, e.g. $(G^R)^{zz}_{xy}$ and $(\pi^R)^{zz}_y$, related to the Rankine component G^R of the Green function and the function π^R defined by (41). Simple analytical expressions for these functions are given elsewhere (due to space limitation).

The Fourier-Kochin approach defines the Fourier component $\widetilde{\phi}^F$ in (38) as a singular double Fourier integral that involves a spectrum function S defined by boundary distributions of elementary waves. For deep water, this Fourier-Kochin representation of $\widetilde{\phi}^F$ is given by (45). The amplitude functions in the spectrum function (45b) are given by (49) for the weight function (48). In the limits $k/k_* \to 0$ or $k/k_* \to \infty$ (for small or large wavenumbers), the amplitude functions (49) are identical to the amplitudes (47a) or (47b) associated with the classical potential representation (3) or the weakly-singular representation (8), respectively (in accordance with the property that the classical and weakly-singular potential representations correspond to farfield and nearfield representations, respectively).

The wave component G^W in expression (50a) for the simple Green function given in *Noblesse and Yang (2004b)* consists of three components W^i, W^+ and W^- that represent distinct wave systems created by a pulsating source advancing at constant speed. The corresponding wave potentials $\widetilde{\phi}^{\,i}$ and $\widetilde{\phi}^{\,\pm}$ in the potential representation (50b) are defined by three nonsingular single (one-fold) Fourier integrals that involve a wave spectrum function given by boundary distributions of elementary waves (45c).

For deep water and in the farfield, this Fourier-Kochin representation of the wave potentials $\widetilde{\phi}^{\,i}$ and $\widetilde{\phi}^{\,\pm}$ in (50b) is given by (52). Expressions (52c) and (52d) for the amplitude functions in the farfield wave spectrum function (52b) correspond to the classical potential representation (3) and the weakly-singular representation (8), respectively. The alternative expressions (52c) and (52d) can be considered optimal for the wave potentials $\widetilde{\phi}^{\,i}$ and $\widetilde{\phi}^{\,\pm}$, respectively, in that they yield minimal numerical cancellations between the components S^W_Γ and S^W_B, associated with the line integral around the curve Γ and the surface integral over the body surface Σ_B, in (53a). In the nearfield, the function Θ_0 in expression (52b) for the farfield wave spectrum function S^W_∞ is replaced by a set of functions that are closely related to the function Θ defined in *Noblesse and Yang (2004b)*. The nearfield wave potentials $\widetilde{\phi}^{\,i}$ and $\widetilde{\phi}^{\,\pm}$ are given elsewhere (due to space limitation).

Both the free-surface terms A_0 and π^ϕ, given by (36b) and (32a), in the potential representation (35) vanish in the farfield. Free-surface integration in (35) therefore is only required over a finite nearfield region of the unbounded free surface Σ_0. Two other (already noted) useful properties of the potential representation (35) are that it defines a continuous potential $\widetilde{\phi}$ at the boundary of the flow domain, and that it does not assume the potential satisfies the usual linearized free-surface boundary condition $\pi^\phi = 0$. Furthermore, the potential representation (35) — with the Green function G chosen as the simple Green function given in *Noblesse and Yang (2004b)* — yields a potential representation for wave diffraction-radiation with forward speed that only involves boundary distributions of elementary free-space Rankine sources and elementary waves (two complementary fundamental solutions of the Laplace equation), and nonsingular single (one-fold) Fourier integrals. This potential representation is significantly simpler than the classical representation that has been used previously in the literature.

References

Hunt B. (1980) *The mathematical basis and numerical principles of the boundary integral method for incompressible potential flow over 3-D aerodynamic configurations*, in *Numerical Methods in Applied Fluid Dynamics*, Academic Press

Noblesse F. (2001) *Velocity representation of free-surface flows and Fourier-Kochin representation of waves*, Applied Ocean Research, 23:41-52

Noblesse F., Yang C. (2002) *Alternative boundary-integral representations of ship waves*, 21st Il Conf. on OMAE, Oslo, Norway

Noblesse F., Yang C. (2004a) *Weakly-singular boundary-integral representations of free-surface flows about ships or offshore structures*, Jl of Ship Research, 48:31-44

Noblesse F., Yang C. (2004b) *A simple Green function for diffraction-radiation of time-harmonic waves with forward speed*, Ship Technology Research, 51:35-52

Noblesse F., Yang C. (2004c) *Wave diffraction-radiation by ships and offshore structures*, 9th Symp. on Nonlinear and Free-Surface Flows, Hiroshima, Japan; pp.1-12